

A RATIONAL ALTERNATIVE FARE STRUCTURE FOR BRITISH RAIL COMMUTERS

A Comment

By Alexander Grey

There are many mistakes in J. G. Gibson's article (Gibson, 1981). First, neither in his discussion of the fares to be charged nor in discussing other issues like seat utilisation and route truncation does he take much account of the complex interaction between the supply of and demand for rail travel—a surprising omission in view of the numerous mathematical models that are available for these purposes. It is on the models that he does use, however, that further comment may be helpful, to remove misunderstandings which might otherwise have arisen.

CONTINUOUS FUNCTIONS

Gibson starts, quite reasonably, by defining the cumulative number of passengers (n) on a train as a continuous function of the train's distance (m) from the terminus (to which they are all, by implication, travelling, though he does not state this explicitly). But he then fails to distinguish sufficiently clearly between the number n who have boarded the train *by* m , and the number who are boarding *at* m which is $-dn/dm$. For example, he says that the number boarding *at* the first station, a distance M from the terminus, is n_1 . But this can be so only if n is defined as a suitable step function of m , and this is contrary to Gibson's intention (he says he is not using step functions).

FARES FUNCTION

There is also considerable confusion over the definition and analysis of the fares function. Gibson seems to be saying that the fare F paid by all passengers boarding *at* m is defined as follows:

$$f = \frac{dF}{dm} = \frac{R}{nM} = \frac{R}{kM} \left(m + \frac{k}{N} \right)$$

This is his equation (3). It seems to be there as a definition, not as a deduction as he supposes. His quadratic fares function (5) can be obtained from (3) by integration, but he does not do this. In fact the argument he uses (on p. 271) to derive (5) is erroneous, based as it is on a misunderstanding of the mathematical concept of a differential—a misunderstanding which leads him to assume that F and dF/dn are the same thing, which they are not.

AXIOMS

Two axioms are stated on p. 270. The first assumes, strangely, an unsubsidised railway, but fortunately it is redundant—he does not use it to constrain his analysis in any way. The second axiom, that the rate at which passengers' fares are changing with distance is the same for all passengers on the train at any time, is meaningless. Passengers pay only one fare, for the distance from the place at which they board the train. Its rate of change with distance can only be defined at that point; it has no meaning for passengers at any later time while they are sitting on the train. Gibson could instead have stated as an axiom that the rate of change of fare with distance is the same for all passengers whenever they board the train. All this would mean is that dF/dm is constant for all m , or that F is a linear function of m , not the quadratic that Gibson supposes.

REVENUE

The calculations of total revenue on p. 272 are mathematically incorrect and logically confused. He is, however, correct in saying that the constant R used in the fares function that he defines is the total revenue from all passengers. This can be proved as follows.

$$\begin{aligned} \text{Total revenue} &= \int_0^M -F \frac{dn}{dm} dm + F_M n_1 \\ &= - \left[Fn \right]_0^M + \int_0^M n \frac{dF}{dm} dm + F_M n_1 \\ &\quad \text{(integrating by parts)} \\ &= \int_0^M \frac{R}{M} dm \\ &= R \end{aligned}$$

A more logical procedure would have been first to define the fares function as $n dF/dm = c$, and then to show that the constant c is equal to R/M , where R is the total revenue.

This approach can be illustrated by keeping Gibson's hyperbolic passenger loading function, but substituting for his fares function a tapered function of the form

$$\frac{dF}{dm} = \frac{c}{m + \frac{k}{N}} = \frac{c}{k} n$$

As before, the total-revenue R from all passengers is given by

$$R = \int_0^M n \frac{dF}{dm} dm$$

which comes to

$$\begin{aligned}
 R &= \int_0^M \frac{ck}{\left(m + \frac{k}{N}\right)^2} dm \\
 &= - \left[\frac{ck}{m + \frac{k}{N}} \right]_0^M \\
 &= c(N-n_1)
 \end{aligned}$$

Thus the tapered fares function can be defined as

$$\frac{dF}{dm} = \frac{R}{(N-n_1)\left(m + \frac{k}{N}\right)}$$

or

$$F = \frac{R}{N-n_1} \left\{ \log \left(m + \frac{k}{N} \right) - \log \frac{k}{N} \right\}$$

A tapered fares function of this kind does neither better nor worse than the one used by Gibson, when considered against the factors discussed in his article. A full analysis of other factors relevant to a proper understanding of the interaction between the demand for and supply of rail travel might, however, tell a different story.

A Rejoinder

By J. G. Gibson

This is a rejoinder not only to the present comment by Grey, but also to an earlier one by Nash (1982).

NASH'S COMMENT

Nash's comment, which might be regarded as a statement of British Rail's views, mainly describes and attempts to justify the Board's current practice. Therefore, as a comment on the model described in my paper, it is mainly irrelevant. But some of his points do merit a reply.

Nash states that my assumptions are inappropriate. To which assumptions does he refer? It cannot be the two axioms which I consider to be the rational base for my

model; these could be called incorrect if, and only if, they were impossible, so perverse as to be virtually impossible, or irrelevant. None of these criticisms applies, therefore the axioms are quite legitimate. The other assumptions concern the variation of train loading with distance and the total cost/mile of running trains. I was obliged to make assumptions about the train loading because of the complete lack of any experimentally determined information. Indeed, it has been stated that such information would be so enormously expensive to obtain as to be impossible (and this in the Age of Information Technology). Thus I chose to assume two cases, one hyperbolic (admittedly a bit unrealistic) and the other linear. The latter is actually used by Nash himself in his discussion. So who is to judge what is appropriate? As far as the assumption about costs/train mile is concerned I consider my estimate of a uniform cost/mile (taking all costs into account) to be reasonably near the truth.

Nash is concerned with discussing a system based on the idea of maximising passenger miles. This can be easily achieved, for example, by setting all fares to zero or by introducing a strong distance taper to encourage long-distance travellers. Such a system is obviously fundamentally different from my model. One way to give Nash's system some semblance of credibility is to manipulate costs. This leads to some strange notions; Nash says that the only mileage-related costs are fuel and rolling stock and track maintenance. Are not the capital costs of the tracks, and the capital, maintenance and staffing costs of the stations which appear at regular intervals along them, also a function of mileage? Surely it would be fundamentally wrong for the principal costs of any transport system to be independent of mileage?

A great deal of woolly and illogical thinking arises from an obsession with marginal costs. There is a tendency to expand the context until, in the end, practically all costs are classified as marginal. Nash's third paragraph purports to be discussing marginal costs; the addition of extra trains can hardly be so classified. In the same paragraph we are asked to believe that two costs, one of which is relatively small compared with the other, should, simultaneously, be virtually the same!

There is an implication in Nash's paragraphs 3 and 4 that I do not recognise the possibility that different train services may start at different distances from Central London. Nothing could be further from the truth. I am quite prepared to apply my model to the case of many train services operating over different distances on multiply branched track.

One final point which is worth note perhaps is that the source of the data given in Nash's Tables 1 and 3 is the British Railways Board. Tables 1 and 2 add nothing useful to the discussion, and Table 3 tells an incomplete story.

GREY'S COMMENT

Grey's comment also falls into two sections—a relevant part which derives mainly from his misreading of my paper, and an irrelevant part which discusses his own models.

I do not imply that all passengers boarding the train are travelling to the terminus. I clearly state in my discussion that the turnover of passengers en route is immaterial provided the overall loading pattern is adhered to.

The condition at the first station M, n_1 is a simple boundary condition on a

continuous curve. Immediately after Equation 4 I make it perfectly clear that dF_m is the total fare received from dn_m passengers. Thus the fare received from each passenger is dF_m/dn_m . The total revenue received from all passengers is the integral of this over the whole journey, and is defined as F_T . The integration shows that $F_T = R$. Grey gets this nomenclature confused and there is no point in discussing it further.

Grey considers two models of his own which I do not regard as relevant comment. The first model, which is not defined very well in terms of its purpose and strategy, results in a linear fare structure. His second model is based on an instantaneous fare rate which is proportional to the number of passengers (n), that is, inversely proportional to distance (m), if a hyperbolic loading is assumed. This, of course, is the opposite sense to my model, and naturally leads him to a final fares function which tapers with increasing distance in a similar sense to current BR practice. To claim that this is neither better nor worse than my proposal seems remarkably contentious. If Grey had begun his comment with this remark, instead of ending with it, the resultant discussion might have been more illuminating.

REFERENCES

- Gibson, J. G. (1981): "A Rational Alternative Fare Structure for British Rail's London and South East Commuter Passengers". *Journal of Transport Economics and Policy*, Vol. XV no. 3 (September), pp. 269–275.
- Nash, C. A. (1982): "A Rational Alternative Fare Structure for British Rail's London and South East Commuter Passengers. A Comment". *Journal of Transport Economics and Policy*, Vol. XVI no. 1 (January), pp. 95–99.