A SIMPLE BUS LINE MODEL FOR OPTIMISATION OF SERVICE FREQUENCY AND BUS SIZE

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INTRODUCTION

This analysis explores the consequences of including certain social costs in the analysis of bus operations. The social costs are passengers' times—waiting, riding, etc. These are monetarised and considered alongside producer costs, so defining passenger inputs as part of the supply price for bus transport. The analysis is restricted to the supply side. The question is how far each particular output of bus transport could be produced at lower social cost, or, in essence, how far bus costs should be traded against passengers' time costs. Since the demand side is left out (that is, customers' reactions to the different supply prices are not dealt with), it will not be possible to determine what the optimal level of supply is in a particular situation. Nevertheless, by means of a simple bus line model it is possible to show that social cost minimisation results in a pattern of service characteristics which is radically different from most present services, mainly in these respects: given the demand, more buses should be run, and the buses should be much smaller. In particular, in off-peak the frequency of service should be substantially higher. Under some conditions it is optimal to run even the same number of buses in off-peak as in peak.

The most likely reason why operators do not perform in this way is simply that they underestimate the user costs. The points made depend critically on the use of realistic time values. The values used in the present analysis are deliberately chosen to be conservative compared with findings of recent econometric studies of travel demand (see for example Bruzelius, 1978). The point-of-departure of the discussion is the contributions made by William Vickrey (1955) and Herbert Mohring (1972), which can be summarised in the "square root formula" for optimisation of service frequency.

OPTIMAL FREQUENCY BY "THE SQUARE ROOT FORMULA"

As a reaction against the "common sense" proposition of making the frequency of service proportional to total patronage, the so-called square root formula adopts the "generalised cost approach". By trading off bus capacity costs against user costs of waiting time, it is found that, given the bus size S, the frequency of service F should be approximately proportional to the square root of the number of passengers carried on a bus line. Using a simple model of a bus line, the square root formula is derived below.

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Consider an urban bus line which starts and ends at a given point. It is largely inconsequential for the analysis whether it is a radial, diametral, or circular route. The "production technique" is simply to make bus round trips by a certain number of buses each holding a maximum of S passengers.

D = round trip distance of the route
R = total round trip *time*
N = number of buses on the route
F = frequency of service (bus flow per hour)
Z = total bus company cost per bus-hour
B = number of passengers boarding buses per hour
Q = average passenger flow per hour; $Q = B(J/R)$, where J is the average journey length
t = boarding and alighting time per passenger
T = running time plus total transitional time of the stops per round trip (transitional time is the time of stopping and starting)
c = value of riding time of passengers
v = value of waiting time of passengers

The total round trip time is the sum of T and the total time of boarding and alighting per round trip. The latter time is the product of t and the number of passengers per bus round trip, B/F.

$$R = T + t\frac{B}{F} \tag{1}$$

The frequency of service F is equal to the product of the density of buses on the route, i.e. the number of buses per km, N/D, and the overall speed, D/R.

$$F = \frac{N}{D} \cdot \frac{D}{T + t\dfrac{B}{F}} = \frac{N - tB}{T} \tag{2}$$

The total cost components to be considered are (1) the bus company costs, (2) the waiting time of passengers, and (3) the riding time of passengers.

The total cost to the bus company per hour equals the product Z and N. Assuming that the mean passenger waiting time at stops is half the bus headway, the total waiting time cost per hour is $vB/2F$, since the headway is the inverse of the frequency. The total riding time cost per hour of the passengers is equal to the product of c, N, and the mean occupancy per bus, which is Q/F. The "balancing factor" in the trade-off between bus company costs and bus passenger costs (that is, the control variable to be optimised) is the number of buses put in on the route, N. The total social costs considered, TC, can be written as a function of N:

$$TC = ZN + \frac{vBT}{2(n - tB)} + \frac{cNQT}{(N - tB)} \tag{3}$$

For each given level of patronage of the bus line, the optimal number of buses on the route N^* is found by setting the derivative of TC with respect to N equal to zero.

$$\frac{\partial TC}{\partial N} = Z - \frac{vBT}{2(N - tB)^2} - \frac{ctBQT}{(N - tB)^2} = 0 \tag{4}$$

$$N^* - tB = \sqrt{\frac{BT\left(\frac{v}{2} + ctQ\right)}{Z}} \tag{5}$$

The optimal frequency of service F^* is obtained by dividing $N^* - tB$ by T (see (2) above).

$$F^* = \sqrt{\frac{B\left(\frac{v}{2} + ctQ\right)}{TZ}} \tag{6}$$

It is seen that F^* will develop at a somewhat greater rate than just proportionally to the square root of the total patronage. However, for low to moderately high values of Q the term $v/2$ will dominate over the second term within the bracket. When Q is comparatively high, a certain limitation of this simple formula becomes more important, namely: the implicit assumption of the square root formula that the total capacity of the buses on the route is always sufficient to meet the demand. The bus size, S, has to be introduced as a control variable as well as the number of buses, N. The typical sizes of the buses currently in use (compare e.g. Table 6) are so relatively large that it seems—from a social point of view—as if their number could be determined with regard only to the frequency of service. The resulting total capacity seems almost always to be sufficient. The reason for this peculiarity is simply that existing buses are oversized from a social point of view. The matter of optimal bus size will be taken up later.

Another question is what is the optimal frequency of a given bus line at different times of the day. It is clear that Z takes radically different values in peak and off-peak. To be able to discuss the optimal frequency at different times of the day, a more penetrating analysis of an urban bus company's cost structure is required. This follows below. Then we can extend the square root formula to different peak and off-peak conditions.

URBAN BUS COMPANY COST STRUCTURE

The following two sections derive peak and off-peak incremental costs of bus transport capacity. Deriving the composition of a bus company's total cost from its accounts is somewhat difficult. In Table 1 the total costs of 21 Swedish bus companies are presented in accordance with the cost classification used by the Association of Swedish Local Bus Service Operators.

A division into the two main categories, (1) traffic operation costs and (2) overhead costs, is obtained by defining overheads to consist of administration, pensions, and all capital costs excluding the capital costs of buses. The remaining costs—the traffic operation costs—are then about 80% of the total costs. This proportion is also

TABLE 1

Composition of the Total Costs of 21 Swedish Urban Bus Companies in 1975 by "Cost Centres"

Cost Centres	*% of Total Cost*
Administration	3.6
Traffic (bus crew costs)	41.8
Workshops and garages (including repair and maintenance of buses)	13.0
Buildings	2.4
Insurance and taxes	3.9
Bus capital costs	20.9
Pensions	7.6
Fuel	6.8
Total	100

Source: Kollektivtrafik i tätort. Bilaga 1. SOU 1975: 48.

obtained in the study of "Costing of Bus Operations" in Bradford 1972–1973 (R. Travers Morgan & Partners, 1974), if the "variable overheads", which include cleaning, lubricating, repairs and maintenance of vehicles, are counted as traffic operation costs. The crew costs make up about one half, and the bus capital cost one quarter, of the total traffic operation costs. The remaining quarter of the traffic operation costs comprises fuel, repairs and maintenance, insurance and taxes, and some other minor items.

A common method of reallocating the total traffic operation costs of a bus company to different lines is to start by allocating the costs according to these three factors:

Total bus-hours operated (H)
Total bus-kilometres (M)
Peak vehicle requirement (N)

It is normally assumed that the total traffic operation costs of any bus line can be reasonably well represented by this linear relationship:

$$TC = aH + bM + cN \tag{7}$$

The main disagreement is about the overhead costs. Can the bus company overheads be allocated at all between different bus lines? If this is possible, should the overheads be apportioned according to the peak vehicle requirements of different lines, or should all the three coefficients a, b and c contain an overhead cost element? A definite solution to this problem does not yet exist.

When it comes to the further division of the costs of a bus line between peak and off-peak traffic, the direct bus running costs, including fuel, lubricants, tyres, and wear and tear, cause no particular problem. The running cost per bus-kilometre can be assumed to be constant, independent of the time of day a certain kilometre is

performed. The remaining traffic operation costs consists of bus "standing costs" and crew costs. The standing cost of a bus per unit of time includes the bulk of the bus capital cost and insurance,[1] the garage cost, licence and other possible taxes. For a bus which is already acquired, the capital cost should be viewed as the opportunity cost of not using the bus on another route where the need for bus transport capacity is at its greatest.

In the following analysis the convenient assumption will be made that this opportunity cost is—as it should be in the normal case—equal to the use-independent capital cost of a new bus. The total bus standing costs are generally assumed to be proportional to the "peak vehicle requirement". This assumption is based on the fact that, in workday schedules of an urban bus line, two principal categories of buses can be distinguished:

1. Buses in all-day service—"all-day buses"
2. Extra buses for peak service—"peak-only buses"

The total number of buses depends entirely on the "peak vehicle requirement", because, if the off-peak frequency of service is to be increased, no additional buses have to be acquired; one simply puts a peak-only bus into all-day service.

Crew costs

It is normally impossible to employ (and pay) drivers and conductors for the actual peak hours only, let alone for just the peak within the peak. Given this fact, an additional problem is that the two diurnal peaks cannot be covered by one straight shift. Second-best solutions are to use double half-day working crews for extra peak services, or to employ "split shifts". In both cases the payable hours are at least twice the work-hours actually required.

In the normal case of marked peakiness of demand, crew scheduling and crew wage rate differentiation with a view to minimising crew costs is very complicated. Actual payment systems are often too complex for a detailed rendering within a standard bus operation costing framework. Different statistical costing methods have been used as a short-cut solution. Arthur Andersen & Co. (1975) have found that a satisfactory explanation of the total crew cost is obtained by a linear relationship where the total bus hours operated in the peak period and off-peak period, respectively, are explanatory variables. It seems that the peak coefficient is about twice the off-peak coefficient.

The study of the cost structure of bus operations in Bradford by R. Travers Morgan & Partners in 1972–1973 included a detailed investigation of crew scheduling and wage costs. The main conclusions were that the basic all-day level of service is maintained by means of two consecutive straight shifts, while the extra peak service required is carried out by split shift working. As a result, the amount of idle time is about the same as the amount of effective working time (bus-hours operated) in split shifts, while the idle time is relatively insignificant for straight shifts. (See their

[1] The economic life (in years) of a bus can be assumed to depend to some extent on the amount of use, i.e. bus-kilometres and/or bus-hours operated. Like insurance, part of the capital costs should therefore appear as an element in the running costs.

diagram 3·2.) This is consistent with the aforementioned finding, that the peak coefficient is about twice the off-peak coefficient of crew hours.

The basic fact is, however, that the input of another bus in peak will cause an incremental crew cost equal to the cost of a complete split shift w', and the input of another bus for all-day service will cause an incremental crew cost which is equal to $2w$, where w is the total cost of a straight shift.

INCREMENTAL COSTS OF PEAK AND OFF-PEAK BUSES

Costs to the bus company

On the basis of the preceding discussion, the incremental costs to a bus company of expanding capacity by the applicable "least unit of capacity" in off-peak and peak, respectively, are obtained in this way:

Off-peak capacity expansion is obtained by putting in another all-day bus making n round trips per day, $n - n_1$ in off-peak and n_1 in peak, and a peak-only bus which makes n_1 round trips is withdrawn;

Peak capacity expansion is obtained by putting in another peak-only bus, making n_1 round trips.

The incremental costs per day in the two cases, IC_0 and IC_1, are:

$$IC_0 = 2w - w' + r(n - n_1)\,D, \text{ provided that } N_0 < N_1. \tag{8a}$$

$$IC_1 = w' + s + rn_1\,D, \text{ provided that } N_1 \geq N_0 \tag{8b}$$

r = running cost per bus-kilometre
n = number of round trips per bus in all-day service
n_1 = number of round trips per peak-only bus
D = round trip distance
s = bus standing cost per day
w = crew cost of straight shift per day
w' = crew cost of split shift per day
N_0 = number of buses in service in off-peak
N_1 = number of buses in service in peak

It may seem far-fetched, but it will nonetheless prove useful, to consider also a situation where there are more buses in service in off-peak than in peak. This "odd case" would imply that some all-day buses stand idle during the peak hours. In this unlikely situation the incremental cost of another bus for off-peak service would be augmented both by adding the standing cost (s), and by not subtracting w'. (There would be no peak-only bus to withdraw.) The incremental cost of another bus for peak-only service would only comprise the running cost rn_1D.

$$IC_0' = 2w + S + r(n - n_1)\,D, \text{ provided that } N_0 \geq N_1 \tag{9a}$$

$$IC_1' = rn_1D, \text{ provided that } N_1 < N_0 \tag{9b}$$

The relative size of IC_0 and IC_1 is a key factor for the present issue of optimal frequency of service in peak and off-peak. In the preceding section it was concluded that the following composition of the total costs of urban bus operation seems to be representative of both Swedish and British conditions:

Crew costs	$4a$
Bus standing costs	$2a$
Bus running costs	a or $2a$
Overheads I	a or –
Overheads II	$2a$

There are different views about the treatment of some costs of repair and maintenance, whether they should be treated as running costs or as a class of overhead costs.

The main problem for the calculation of the relative size of IC_0 and IC_1 is that the ratio of IC_0/IC_1 is quite sensitive to the way of dealing with the overhead costs. R. Travers Morgan & Partners (RTM) (1974) have applied a "marginal costing" approach to give incremental costs per bus for peak and off-peak services, respectively. It involves the attribution of all overhead costs (overhead I and II) to the peak vehicle requirement. A different procedure has been adopted by the National Bus Company (NBC), which first allocates certain "semi-variable" costs per bus-hour, and secondly divides the "fixed" overheads into two parts: one part is attributable to the peak vehicle requirement, and the other part is attributed to bus-hours. A third method of incremental cost calculation would be simply to disregard overheads like "general management" as well as other administrative costs which are particularly difficult to allocate, while adhering to the NBC treatment of the "semi-variable" costs.

By applying these three methods of dealing with the overhead costs, the relative sizes of IC_0 and IC_1 are calculated, assuming that the ratio of the two peak periods to the total off-peak period is 4 : 10. A further assumption is that the crew cost per split shift is about 10% higher than the crew cost per straight shift.

As is seen from Table 2, by the RTM method the IC_0/IC_1 ratio comes to 1/3, while the NBC method gives a good 2/3. Leaving the overheads out of consideration altogether makes the incremental cost ratio equal to 2/3.

A corresponding incremental cost calculation is based on the assumption that $N_0 > N_1$, yields values for the ratio of IC_0'/IC_1' of 3/100 by the RTM method, and about 1/10 by the other two methods.

Reduction in costs to passengers

The reduction in costs to passengers from putting in another bus is calculated by taking the derivative of the total waiting and riding time costs (see (3) above) of the passengers in peak and off-peak, respectively, with respect to N_0 and N_1. The resulting reductions, IB_0 and IB_1, are:

$$IB_0 = \frac{E_0 B_0 T_0 \left(\frac{v}{2} + ctQ_0\right)}{(N_0 - tB_0)^2} = \frac{E_0 B_0 \left(\frac{v}{2} + ctQ_0\right)}{T_0 F_0^2} \tag{10a}$$

$$IB_1 = \frac{E_1 B_1 T_1 \left(\frac{v}{2} + ctQ_1\right)}{(N_1 - tB_1)^2} = \frac{E_1 B_1 \left(\frac{v}{2} + ctQ_1\right)}{T_1 F_1^2} \tag{10b}$$

TABLE 2

Incremental Costs of Off-Peak and Peak Buses, Applying Three Different Principles of Overhead Costs Allocation

Cost Item	*RTM*		*NBC*		*Disregard Overheads*	
	IC_0	IC_1	IC_0	IC_1	IC_0	IC_1
Crew cost	1.8*a*	2.2*a*	1.8*a*	2.2*a*	1.8*a*	2.2*a*
Standing cost	—	2.0*a*	—	2.0*a*	—	2.0*a*
Running cost	.7*a*	.3*a*	1.4*a*	.6*a*	1.4*a*	.6*a*
Overhead cost	—	3.0*a*	.9*a*	1.1*a*	—	—
Total incremental cost	2.5*a*	7.5*a*	4.1*a*	5.9*a*	3.2*a*	4.8*a*

where

E_0 = extent of the off-peak periods per day

E_1 = extent of the peak periods per day

$E = E_0 + E_1$

With a reference to the list of notations given above the rest of the symbols are self-explanatory

The crucial point for the present discussion is that, although the rate of peak demand is substantially higher than the rate of off-peak demand, the total number of passengers travelling in off-peak can be as large as, or even larger than, the total number of peak passengers. This tends to make IB_0 and IB_1 of a comparable order of magnitude for each given frequency of service.

The proportion of total peak and off-peak travel varies certainly from one town to another, but some typical values can be given. The peak period per day, E_1, is typically about 4 hours—2 hours in the morning and 2 hours in the afternoon. The "all-day" level of service may be maintained from say 6.00 to 20.00. (After that a night service may be run; that, however, is outside the scope of this paper.) In that case $E_0 = 2.5E_1$.

The number of trips per off-peak hour, B_0, can be assumed to be from 1/3 to 1/2 of B_1 (remember that the directional imbalance is normally substantially greater in the peak period). Combining these two assumptions gives as a result that the ratio E_0B_0/E_1B_1 will range from 0.83 to 1.25. It seems, however, that it is more commonly above than below unity. In Bradford, for example, the total number of trips made in the four peak hours is 43% of the total number of trips per day, according to the Bradford Bus Study.

It is clear that the value of the factor within the brackets of the numerators of (10a) and (10b) is somewhat lower in off-peak than in peak, since $Q_0 < Q_1$. It can, however, be shown that this difference is of minor importance for the relative size of the whole value of the factor within the bracket. Not until the passenger flow is unrealistically dense on a particular route will the difference be significant. There are four parameters involved: the value of waiting time, v, the value of riding time, c, the boarding and

TABLE 3
Low and High Extreme Values of the Ratio $(v/2 + ctQ_0)/(v/2 + ctQ_1)$

Q_1	*Low Value*	*High Value*
50	0.97	0.99
100	0.95	0.97
150	0.93	0.96
200	0.91	0.95
250	0.89	0.94
500	0.81	0.88
1 000	0.71	0.81

alighting time per passenger, t, and the ratio of Q_0 to Q_1, which is denoted α. A sensitivity analysis is carried out by giving "high" and "low" values to each of these parameters. Then the ratio of $(v/2 + ct\alpha Q_1)/(v/2 + ctQ_1)$ is calculated for different levels of the peak passenger flow in two extreme cases: in one case v and α are given the high values, while c and t are given the low values, and in the other case it is the other way round. The parameter values used are:

	Low value	*High value*
v	£1	£2
c	£1/3	£2/3
t	2.15 sec.	4.25 sec.
α	1/3	1/2

It may be mentioned that the "low value" of t is applicable to two-man bus operation, and the "high value" to one-man operation. See Cundill and Watts (1973) and Quarmby (1974).

In conclusion, it thus seems that, for one and the same frequency of service, the reduction in passengers' costs from the running of another bus is more or less the same in peak and off-peak. In Figure 1 this condition is taken into account by giving the same shape to the two functions (IB_0 of N_0 and IB_1 of N_1).

THE CASE FOR RUNNING THE SAME NUMBER OF BUSES IN PEAK AND OFF-PEAK

By balancing the incremental passenger and bus costs of another bus in each period, the optimal number of buses could presumably be obtained. In the diagrams of Figure 1 IC_0 and IC_1 are put in, assuming that the former is two-thirds of the latter. The point is now that the resulting values for the number of buses in off-peak and peak service, N_0^* and N_1^*, do not fulfil the precondition for the relative level of IC_0 and IC_1, that is $N_0 < N_1$. If the RTM costing method were applied, the positive difference between N_0^* and N_1^* would be much larger still.

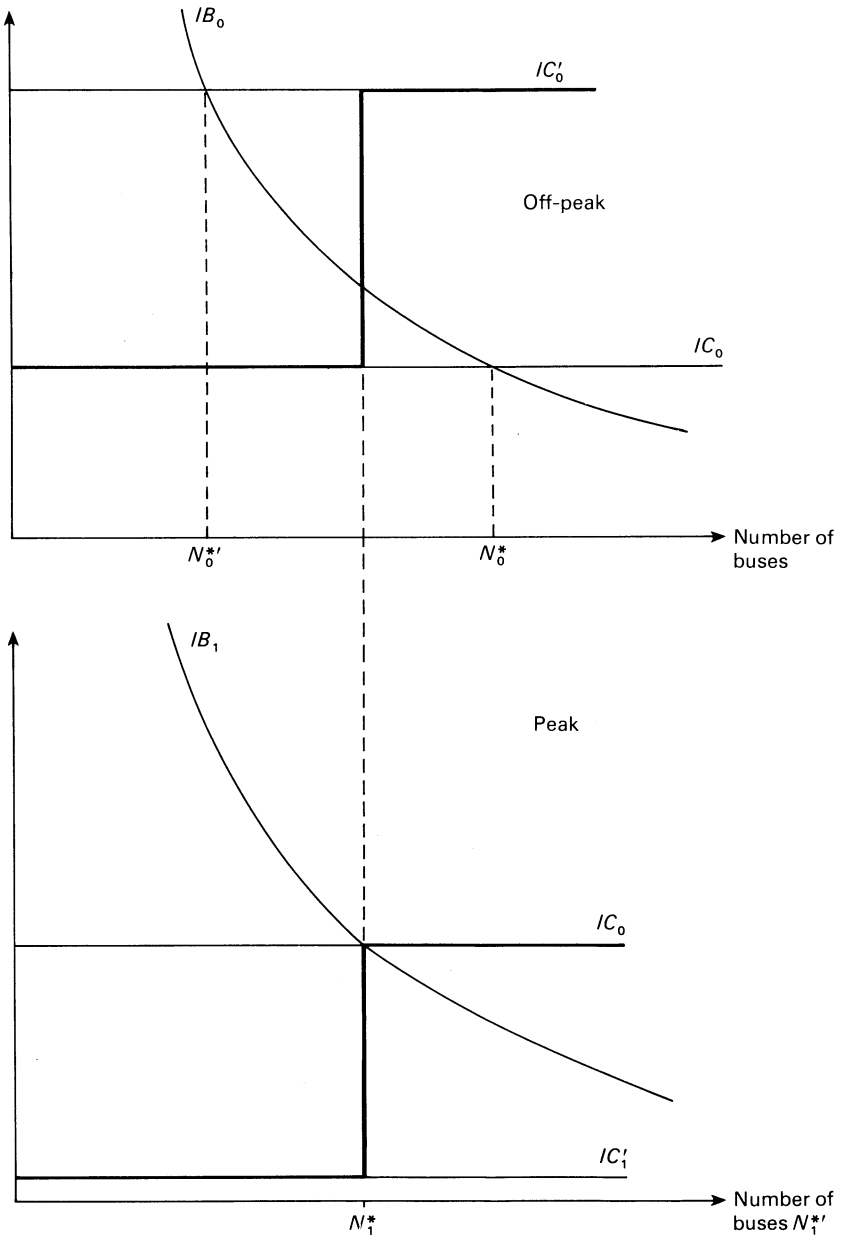

FIGURE 1
Incremental Bus Costs and Reductions in Passenger Costs from an Additional Bus

Applying the value of the ratio IC'_0/IC'_1 would, as expected, not resolve the inconsistency. The result would be that $N_0^{*\prime}$ is much smaller than $N_1^{*\prime}$, which, however, is incompatible with the "perverse" precondition for the relative size of IC'_0 and IC'_1.

These relationships indicate that a "corner solution" is applicable. Given the discontinuous character of the incremental costs, the optimality condition is not necessarily that the incremental cost and benefit should be equal, but that the inequalities should be as follows:

$$IC_0 \leq IB_0 \leq IC'_0 \tag{11a}$$

$$IC'_1 \leq IB_1 \leq IC_1 \tag{11b}$$

Under the conditions given in Figure 1 that $N_0^* > N_1^*$ and $N_0^{*\prime} < N_1^*$, the only alternative which is consistent with (11a) and (11b) is that $N_0^{opt} = N_1^{opt}$, i.e. that the same number of buses is run in peak and off-peak. Diagrammatically this result can be illustrated by writing IC_0 and IC_1 as "step-functions", where the discontinuity occurs when N_0 and N_1 are equal. The position of each of the corresponding incremental benefit functions is such that the point of intersection occurs at the vertical leg of the step.

Needless to say, exact identity of the peak and off-peak incremental benefit functions is not a necessary condition for the result that $N_0 = N_1$ in optimum. The crucial condition is apparently that $N_0^* \geq N_1^*$, which in turn is determined by the relative value of the ratio in peak and off-peak of the total waiting time costs at stops (of passengers on the bus as well as of passengers waiting for the bus) to the incremental cost of another bus.

The fact that the same number of buses is run all day does not necessarily mean that the frequency of service has to be the same, too. It would normally be possible to produce a slightly greater total mileage per hour by a given number of buses in off-peak than in peak. Given that $B_0 < B_1$, the frequency of service could be slightly higher in off-peak than in peak, because fewer passengers are boarding/alighting per stop in off-peak. If in addition $T_0 < T_1$, this effect would be reinforced. It is not absolutely necessary to let this difference appear in the schedule. It is possible to maintain the same frequency in peak and off-peak by means of a more "relaxed" way of driving in off-peak. In off-peak one can afford to be patient with the old lady, to willingly help the mother with the pram, to avoid abrupt stopping and starting, etc. The question is whether this equalisation of the schedule is desirable? It is not a very crucial matter. However, it is not unimportant that a schedule which is uniform all day is easier to remember. If in addition a truly reliable service can be offered, it is quite possible that the mean waiting time at stops can be reduced below the assumed "half the headway" value.

It must finally be pointed out that $N_0^{opt} = N_1^{opt}$ is not a universally valid prescription. Such parameter value constellations as will make $N_0^* < N_1^*$ are certainly not inconceivable. For example, where the off-peak rate of demand is unusually low relative to the peak rate of demand, the frequency of service should most likely be higher in peak than in off-peak. However, a related point to the previous one is that in a situation where N_1^* exceeds N_0^* by a relatively small margin (as it can be expected to do in all but very exceptional cases) there is still a good case for equalising the peak

and off-peak service frequency. As mentioned, this has appreciable inherent advantages, which should not be sacrificed until N_1^* exceeds N_0^* substantially. The simplicity afforded by a standard all-day schedule has a considerable value in its own right.

A MODIFIED VERSION OF THE SQUARE ROOT FORMULA

A new version of the square root formula for the optimal frequency is obtained by the following modified model: on the assumption that not only $N_0 = N_1$, but also that the total round trip time is the same in off-peak and peak, so that $F_0 = F_1$, the total social costs per day can be written (compare (3) above):

$$TC = N \cdot IC + \frac{vEB}{2F} + \frac{cEQN}{F} \tag{12}$$

The frequency of service that N buses can achieve is determined by the peak conditions as to traffic congestion and the rate of boarding and alighting.

$$F = \frac{N - tB_1}{T_1} \tag{13}$$

Inserting this expression for F In (12) and taking the derivative of TC with respect to N gives:

$$\frac{\partial TC}{\partial N} = IC - \frac{vEBT_1}{2(N - tB_1)^2} - \frac{ctEQB_1T_1}{(N - tB_1)^2} \tag{14}$$

Setting this equal to zero and solving for N (observing that $QB_1 = Q_1B$) yields:

$$(N - tB_1)^2 = \frac{EB\left(\frac{v}{2} + ctQ_1\right)T_1}{IC} \tag{15}$$

The optimal frequency is finally found by dividing by T_1^2 and taking the square root of the resulting expression.

$$F^{opt} = \sqrt{\frac{EB\left(\frac{v}{2} + ctQ_1\right)}{IC \cdot T_1}} \tag{16a}$$

That is to say, when the diurnal peaks are taken into account, the original square root formula (6) holds good, provided that the demand per hour, B, is interpreted as a weighted average, i.e. $B = (E_0B_0 + E_1B_1)/E$, and that the value of the peak flow, Q_1, is used in the formula, and provided also that the incremental cost of another all-day bus (IC divided by the total extent of a service day) E is substituted for the previously used unspecified "cost per hour", Z.

A useful variant of this formulation is obtained by expressing B in terms of the peak flow Q_1, and eliminating the specific route characteristic T_1. We introduce the

TABLE 4
Parameter Values Used

Parameter	*Description*	*Value Used*
h	Running and transitional time per km	2.8 min
J	Average journey length	3 km
α	Ratio of off-peak to peak passenger flow	0.4
E_0	Extent of off-peak periods	10 h
E_1	Extent of peak periods	4 h
t	Boarding/alighting time per passenger	4.25 sec
c	Value of riding time	50 p
v	Value of waiting time	150 p
IC	Incremental cost per work-day of all-day bus[a]	
	$S = 45$	£56
	$S = 60$	£61.50
	$S = 75$	£67

[a] Source: *Commercial Motor Journal.* London, 1975.

designation β for the ratio of the mean rate of passenger flow of the whole service day Q to the peak flow Q_1.[2] The mean number of passengers B can then be written as $\beta Q_1 D/J$. A more general route characteristic than T_1 is the ratio of T_1/D, that is, the running time and transitional time *per kilometre.* Designating $T_1/D = h$, the optimal frequency can alternatively be written:

$$F^{opt} = \sqrt{\frac{EQ_1\left(\frac{v}{2} + ctQ_1\right)}{IC \cdot hJ}} \tag{16b}$$

Example of optimisation of service frequency

It is interesting to consider a concrete example of the optimal frequency of a bus service. Assuming the route characteristics and "factor prices" given in Table 4, the previously derived formula for F^{opt} gives as a result the pattern shown in that Table (compare Table 5).

For as moderate a level of flow as 50 passengers per peak hour, a frequency of service corresponding to a bus every ten minutes should be offered, and a bus every five minutes is right for a mean passenger flow of about 200 passengers per peak hour. A bus every third minute should be running when this figure is doubled. Much higher levels of demand are outside the interesting range. When a frequency of service of 25–30 buses per hour is approached, further increases in the density of demand should under normal conditions be met mainly by increasing the density of bus lines (by making the bus line network more fine-meshed). At any rate, the values of the optimal frequency given in Table 5 are well above what is currently offered by urban bus companies.

[2] We have previously used α for the ratio of Q_0/Q_1. It consequently follows that $\beta = (\alpha E_0 + E_1)/E$.

It can be seen that the optimal frequency is relatively insensitive to bus size. In Table 5 three different sizes are compared. It is shown that the differences in the values of F^{opt} for each level of passenger flow are quite insignificant.

Like the original version, the modified version of the square root formula for optimal frequency assumes that the input of buses in peak as well as in off-peak can be determined solely by frequency of service considerations. For the most typical sizes of buses currently in use for urban service, it seems that this important prerequisite holds good generally in the interesting range of demand. This is exemplified by comparing the figures for F^{opt} with the corresponding figures for the minimum frequency of service required for reasons of capacity alone.

The capacity requirement is represented by the condition that a certain value of the mean occupancy rate must not be exceeded. The mean occupancy rate, ϕ, is defined as the ratio of the mean passenger flow to the product of the bus size and the service frequency.

Given the stipulated maximum value of the occupancy rate, ϕ_{max}, and the bus size, S, each capacity constraint defines a minimum permissible service frequency, F_0^{min} and F_1^{min}.

$$F_0^{min} = \frac{Q_0}{\phi_{max} S} \tag{17a}$$

$$F_1^{min} = \frac{Q_1}{\phi_{max} S} \tag{17b}$$

What can be assumed about the value of ϕ_{max}? Two characteristics of, in particular, radial bus lines make the mean occupancy rate comparatively low even in the busiest hours, namely: (i) the tidal flow pattern of demand, which gives rise to a marked directional imbalance in the short run, and (ii) the spatial peak. Only in the "critical section" on the main haul are the buses likely to be full or nearly full, and the length of the critical section is normally only a fraction of the length of the main haul. In Bradford, for instance, it was found that the mean occupancy rate in the four peak hours was only 0.31. For the present exemplification it is assumed that $\phi_{max} = 1/3$. Three bus sizes have been considered, namely, buses holding 45, 60 and 75 passengers. The first figure is rather low by current standards. The middle figure is a typical bus size for British double-deckers. So far as Swedish conditions in larger towns are concerned, this figure is on the low side in spite of the fact that only one-man operation is used. For example, the bus company serving greater Stockholm ("SL") had in 1975 the composition of its bus fleet shown in Table 6.

As is shown in Table 5, with buses holding just 45 passengers, the transport capacity resulting from optimising the service frequency would be sufficient for the peak demand up to a level of passenger flow of about 200 passengers per hour. However, by using the typical Stockholm bus size of 75, the peak passenger flow has risen to 600 before the capacity rather than the frequency of service becomes the determining factor. An indication that this level of the passenger flow represents an unrealistically dense bus route is that no less than 24 buses an hour are required to meet such a demand.

An obvious reflection arising from these findings is that it would be unnecessarily wasteful to use oversized buses. When a social point of view is assumed, one should

TABLE 5
"Optimum" vs. Minimum Frequency of Service

	$S = 45$		$S = 60$		$S = 75$	
Q_1	F^{opt}	F^{min}	F^{opt}	F^{min}	F^{opt}	F^{min}
25	4.4	1.7	4.2	1.3	4.1	1
50	6.3	3.3	6	2.5	5.8	2
75	7.8	5	7.5	3.8	7.2	3
100	9.1	6.7	8.7	5	8.3	4
150	11.4	10	10.9	7.5	10.4	6
200	13.4	13.3	12.8	10	12.2	8
250	15.2	16.7	14.5	12.5	13.9	10
300	16.9	20	16.2	15	15.5	12
400	20.2	26.7	19.3	20	18.5	16
500	23.2	33.3	22	25	21.2	20
600	26.2	40	25	30	23.9	24

TABLE 6
Composition of the SL Bus Fleet

Bus Holding Capacity (No. of Passengers)	*Number of Buses*	%
11–20	5	0.3
21–30	0	0
31–40	4	0.2
41–50	0	0
51–60	8	0.5
61–70	151	9
71–80	1,187	71
81–90	215	13
91–100	0	0
101–110	34	2
121–130	71	4

not content oneself with optimising the number of buses. A further improvement would be to adjust the bus size so that the capacity constraint becomes binding whatever the level of peak flow happens to be.

It is true that only moderate cost savings will be obtained by adjusting the bus size so that the minimum permissible frequency and the optimal frequency are equal in the whole range. The crew cost is the single dominating cost item, and the cost of the driver will obviously not be influenced by the bus size.[3]

To get an idea of the magnitude of the possible cost savings, let us examine the relationship between bus standing and running costs and bus size.

[3] In cases where two-man operation applies, making the bus size a variable may under certain circumstances indicate a change over to one-man operation. In that case the elimination of the cost of the conductor does, of course, not constitute a net saving. Disbenefits in the form of longer time at stops have to be deducted.

BUS COSTS AND BUS SIZE

The *Commercial Motor Journal* (London) regularly gives tables of standing costs and running costs of buses in the range from 12 to 86 seats. The data plotted in Figures 2a and 2b apply to 1975. As is seen, a linear relationship seems to apply in both cases. The economies of bus size are particularly pronounced in running costs. The running cost per mile is not even doubled between the smallest and largest bus in the sample. It can be noted that equally significant economies of bus size would be apparent in the remaining traffic operation costs, if the crew costs were added to the standing costs given in Figure 2b. By least squares regression these relationships were obtained:

$$\text{Running cost per mile (pence)} = 11 + 0.14S \quad (r^2 = 0.94) \tag{18}$$

$$\text{Standing cost per week (£)} = 6.5 + 0.72S \quad (r^2 = 0.98) \tag{19}$$

It thus appears that reducing the bus size by, say, 50% would save a good 25% of the running and standing cost per bus. Bearing in mind that the crew cost (of one-man-operated buses) is size-independent, the cost savings of this size reduction would be only 12–15% of the total traffic operation costs. However, the adjustment of holding capacity to need is not a negligible source of improvement in efficiency in view of the present excessive bus sizes.

OPTIMAL BUS SIZE

When the bus size, S, is assumed to be a control variable, the formula for the optimal frequency (16) is no longer strictly applicable, since the incremental cost IC is a function of S. Moreover, the question is whether the case for running the same number of buses in peak and off-peak remains equally strong when more adequately sized buses are employed?

In what follows the social cost analysis of a bus line is extended by assuming the bus size to be a control variable as well as the number of buses on the route. The addition of this variable makes the analysis somewhat more complicated. The previous informal way of reasoning is no longer tenable. A formal Kuhn-Tucker analysis has to be resorted to.

The total social costs are, as before, assumed to include the total waiting and riding time costs of the passengers, and the total bus running, standing, and crew costs.

As the evidence suggests a linear relationship between the bus size and both the running cost per kilometre ($r = a_r + b_rS$) and the standing cost per unit of time ($s = a_s + b_sS$), it is possible to write the total incremental cost per bus in all-day service and peak-only service, respectively, in abbreviated form as follows:

$$IC = a + bS, \text{ and } IC_1 = a_1 + b_1S \tag{20}$$

where

$$\begin{aligned} a &= na_rD + a_s + 2w, \quad \text{and} \quad b = nb_rD + b_s \\ a_1 &= n_1a_rD + a_s + w', \quad \text{and} \quad b_1 = n_1b_rD + b_s \\ a_0 &= (n - n_1)a_rD + 2w - w', \quad \text{and} \quad b_0 = (n - n_1)b_rD \end{aligned}$$

The total social costs are then written:

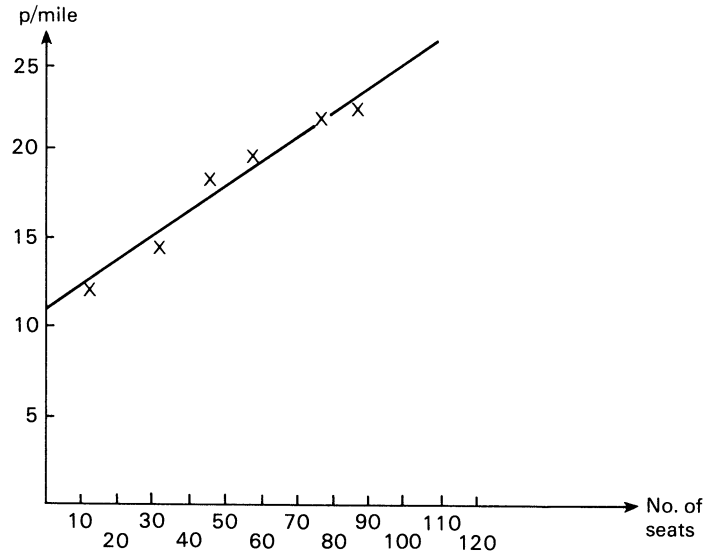

FIGURE 2a
Bus Running Cost

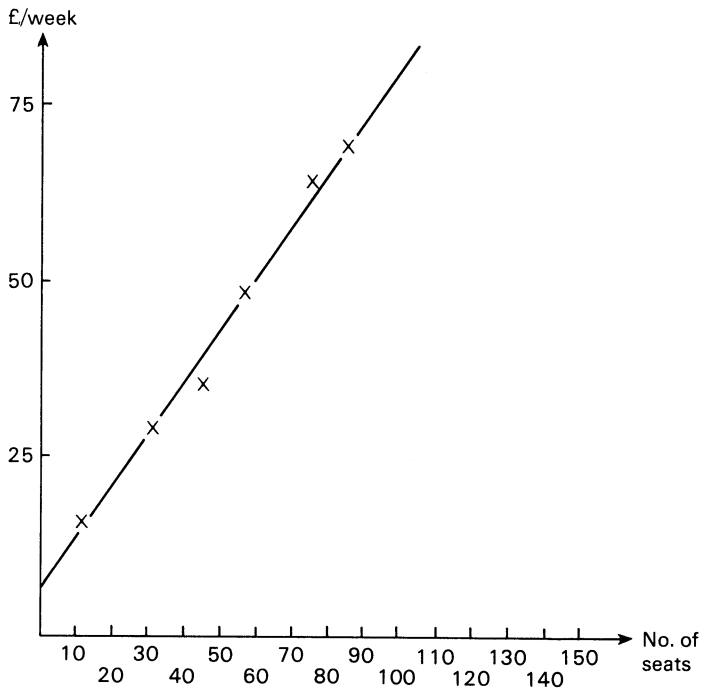

FIGURE 2b
Bus Standing Cost

$$TC = (a + bS)N_0 + (a_1 + b_1S)(N_1 - N_0) + \\ + \frac{\frac{v}{2}E_0B_0T_0 + cN_0E_0Q_0T_0}{N_0 - tB_0} + \frac{\frac{v}{2}E_1B_1T_1 + cN_1E_1Q_1T_1}{N_1 - tB_1} \quad (21a)$$

As before the following inequality applies:

$$N_1 - N_0 \geq 0 \quad (21b)$$

On the other hand, now that the bus size is a control variable it is assumed that the capacity constraint is always binding in peak, while it may or may not be binding in off-peak.

$$\phi_{max} = \frac{Q_1T_1}{S(N_1 - tB_1)} \quad (21c)$$

$$\phi_{max} \geq \frac{Q_0T_0}{S(N_0 - tB_0)} \quad (21d)$$

As S is a function of N_1 (according to the peak capacity constraint), the total social cost can consequently be regarded as a function of just N_0 and N_1.

The applicable Lagrangian expression takes this shape:

$$\pi = TC - \lambda(N_1 - N_0) - \mu_0\left[1 - \frac{Q_0T_0(N_1 - tB_1)}{Q_1T_1(N_0 - tB_0)}\right] \quad (22)$$

Given that both N_0 and N_1 are positive, the Kuhn-Tucker conditions for a minimum can be abbreviated to:

$$\frac{\partial\pi}{\partial N_0} = a_0 + b_0S - \frac{E_0B_0\left(\frac{v}{2} + ctQ_0\right)T_0}{(N_0 - tB_0)^2} + \lambda - \mu_0\frac{Q_0T_0}{S(N_0 - tB_0)^2} = 0 \quad (23)$$

$$\frac{\partial\pi}{\partial N_1} = b_0N_0\frac{\partial S}{\partial N_1} + a_1 + b_1S + b_1N_1\frac{\partial S}{\partial N_1} - \frac{E_1B_1\left(\frac{v}{2} + ctQ_1\right)T_1}{(N_1 - tB_1)^2} - \lambda + \\ + \mu_0\frac{Q_0T_0}{S^2(N_0 - tB_0)}\frac{\partial S}{\partial N_1} = 0 \quad (24)$$

$$\frac{\partial\pi}{\partial\lambda} = N_1 - N_0 \geq 0 \quad (25)$$

$$\lambda\frac{\partial\pi}{\partial\lambda} = 0 \quad (26)$$

$$\frac{\partial\pi}{\partial\mu_0} = 1 - \frac{Q_0T_0(N_1 - tB_1)}{Q_1T_1(N_0 - tB_0)} \geq 0 \quad (27)$$

$$\mu_0 \frac{\partial \pi}{\partial \mu_0} = 0 \tag{28}$$

Let us first examine the case of $N_0 < N_1$. In that case $\lambda = 0$. Two sub-cases are then distinguished: in one the off-peak capacity constraint is binding, in the other it is not. In the sub-case where the off-peak capacity constraint is not binding, $\mu_0 = 0$. Unfortunately, it is not possible to produce an explicit solution for the optimal design in this sub-case. However, a good idea how the case for running the same number of buses in peak and off-peak stands can be obtained in the following way. From (23) the optimal off-peak frequency can be written:

$$F_0^2 = \frac{(N_0 - tB_0)^2}{T_0^2} = \frac{E_0 B_0 \left(\frac{v}{2} + ctQ_0\right)}{(a_0 + b_0 S)T_0} \tag{29}$$

This says the same as has been said previously (when the bus size was assumed to be given), namely, that the optimal frequency is found where the incremental cost ($a_0 + b_0 S$) is equal to the incremental benefit of another bus in off-peak. A corresponding expression for the peak period is obtained from (24), observing that

$$\frac{\partial S}{\partial N_1} = \frac{S}{N_1 - tB_1}$$

$$F_1^2 = \frac{(N_1 - tB_1)^2}{T_1^2} = \frac{E_1 B_1 \left(\frac{v}{2} + ctQ_1\right)}{\left[a_1 + b_1 S - \frac{b_0 N_0 + b_1 N_1}{N_1 - tB_1} S\right] T_1} \tag{30}$$

The right-hand term within the bracket of the denominator comes close to bS under all realistic conditions. The value of the denominator of (30) is consequently approximately equal to $(a_1 - b_0 S)T_1$. The optimal frequency of peak service is thus equal to the ratio of the total costs of waiting time and boarding/alighting time in peak to $a_1 - b_0 S$ (rather than to $a_1 + b_1 S$, which is applicable when S is given), while the optimal frequency of off-peak service is equal to the ratio of the total costs of waiting time and boarding/alighting time in off-peak to $a_0 + b_0 S$. This means that the previous inconsistency—the finding that the "optimal" frequency should be higher in off-peak than in peak—is much less likely to arise. The treatment of the overhead costs is the crucial factor for this issue. If the overheads are allocated per bus-hour or bus-kilometre, or simply disregarded, the result will under normal conditions be that the peak frequency should be somewhat higher than the off-peak frequency. On the other hand, if the overheads are allocated per bus (in accordance with the RTM method) to boost a_1, the case for running the same number of buses all day would remain under ordinary circumstances.

In the former case, it should also be checked that the resulting value of the optimal off-peak frequency is higher than the minimum permissible frequency. In other words, the ratio of F_0/F_1 must be higher than $\alpha\,(= Q_0/Q_1)$; otherwise the capacity constraint is binding in the off-peak, too, and the optimal values of the control variables have to

be recalculated on the assumption that $\mu_0 > 0$. The squared ratio of the off-peak to the peak frequency is obtained by dividing (29) by (30). The result (31) should exceed α^2 to ensure that the solution for the optimal off-peak frequency will not violate the capacity constraint.

$$\frac{E_0 B_0 \left(\frac{v}{2} + ctQ_0\right) T_1 (a_1 - b_0 S)}{E_1 B_1 \left(\frac{v}{2} + ctQ_1\right) T_0 (a_0 + b_0 S)} \geq \alpha^2 \tag{31}$$

It has previously been pointed out that α seems to be in the range of 1/3 to 1/2. It appears safe to conclude that the ratio above is normally well above the upper limit of α^2, that is 1/4.

As has been shown, the case for equalising the service frequency is much weaker when the bus size is assumed to be a control variable. Let us nevertheless consider the "minority case" where it is still optimal to run the same number of buses in peak and off-peak. In this case explicit solutions for the control variables are easily obtained. The value for the optimal bus size should be fairly representative also for the main case. In the main case it may be optimal to make N_0 10–30% lower than N_1. This should, however, not make a very great difference as regards the optimal bus size. The following discussion should therefore have a more general interest than may at first appear, so far as the analysis of the determinants of the optimal bus size is concerned.

Setting $N_0 = N_1 = N$, and summing (23) and (24), eliminates λ and gives:

$$a - \frac{btB_1Q_1T_1}{\phi_{max}(N - tB_1)^2} - \frac{E_0B_0T_0\left(\frac{v}{2} + ctQ_0\right)}{(N - tB_0)^2} - \frac{E_1B_1T_1\left(\frac{v}{2} - ctQ_1\right)}{(N - tB_1)^2} = 0 \tag{32}$$

Assuming as before that the frequency of service is equalised throughout the service day when $N_0 = N_1$, the following expression is obtained for the optimal all-day frequency of service:

$$F^{opt} = \sqrt{\frac{EB\left(\frac{v}{2} + ctQ_1\right) + \frac{btB_1Q_1}{\phi_{max}}}{aT_1}} \tag{33}$$

Comparing this result with the modified square root formula for the optimal service frequency (16a) which assumes a given bus size, two differences stand out. In the numerator of (33) the term btB_1Q_1/ϕ_{max} has been added, and in the denominator a has been substituted for IC. Both differences work to raise F^{opt} in comparison with the situation where the bus size is given. The addition to the numerator is not usually very significant. It has previously been shown that the right-hand term within the bracket, that is ctQ_1, constitutes only a fraction of the left-hand term $v/2$ for moderate flows. A comparison of $EBctQ_1$ and btB_1Q_1/ϕ_{max} shows in turn that the latter term is only a fraction—about one third—of the former. The denominator has decreased, since only the size-independent part of IC—that is a—appears in the denominator ($IC = a + bS$). This means that relaxing the assumption that S is given results not only in

the elimination of all peak excess capacity, but also in a higher service frequency at each level of demand (provided that the capacity constraint is not binding in the original situation). If the existing bus size is rather large in the situation where S is given, the difference between IC and a, that is bS, is relatively significant.

The optimal bus size follows immediately from (33) and the capacity constraint.

$$S^{opt} = \frac{Q_1}{\phi_{max} F^{opt}} = \frac{Q_1}{\phi_{max}} \sqrt{\frac{aT_1}{EB\left(\frac{v}{2} + ctQ_1\right) + \frac{btB_1Q_1}{\phi_{max}}}} \tag{34a}$$

It is useful for a discussion of the determinants of the optimal bus size to rewrite (34a) by expressing B and B_1 in terms of Q_1, the journey length J, and the inverse of the running speed which is assumed to be outside the control of the bus company. We then have:

$$S^{opt} = \frac{1}{\phi_{max}} \sqrt{\frac{ahJQ_1}{\frac{\beta Ev}{2} + tQ_1\left(\beta Ec + \frac{b}{\phi_{max}}\right)}} \tag{34b}$$

In view of the fact that the left-hand term completely dominates the denominator, it follows that S^{opt} is roughly (but not quite) proportional to the square root of Q_1 in the interesting range of demand.[4] Examples of values of S^{opt} for different levels of the peak passenger flow are given in Table 7. The strongest influence on S^{opt} comes, however, from ϕ_{max}. The more unbalanced a route is, and the higher the spatial peak of the critical section rises above the mean level of flow, the lower ϕ_{max} has to be set, and as a result the larger the optimal bus size will be. S^{opt} is nearly inversely proportional to ϕ_{max}. The reason for this relationship is basically that the frequency of service will become less and less important as fewer passengers are carried on the back haul, and outside the critical section on the main haul, relative to the maximum flow. Under conditions of extreme spatial peakiness of demand, the capacity requirement can be profitably met mainly by increasing the bus size. The same goes for the influence of β on S^{opt}. The smaller the number of passengers in off-peak is relative to the peak flow—which determines the capacity requirement—the larger the optimal bus size will be. The journey length, J, plays a similar role. The further each passenger travels on average, the smaller the number of trips (B) will be for a given level of passenger flow (Q); and, again, the capacity requirement is met by bus size rather than frequency.

The influences of the "factor prices" involved, a, b, c, and v, are as expected. The optimal bus size is positively correlated to the size-independent bus cost component, a, and is negatively (although quite weakly) correlated to the proportionality constant b of the size-proportional bus cost component. An increase in the passenger time values c and v works to reduce the optimal bus size (and to raise the service frequency to a corresponding degree). It can be noted that if the crew wage cost and the time values increase roughly in parallel in the future, the combined effect on S^{opt} will, *ceteris paribus*, be nearly nil.

[4] As Q_1 goes to infinity, S^{opt} approaches an upper limit. Using the parameter values assumed in the previous discussion, the upper limit of S^{opt} comes to 95 passengers.

TABLE 7
Social Cost Comparison of Three Different Bus Service Designs

	Optimal Design		*Social Costs[a] per Passenger*		
Q_1	F^{opt}	S^{opt}	AC^{opt}	$AC^{opt}_{S=75}$	$AC^{\phi=\phi max}_{S=75}$
25	5	15	32.9	40.2	136.9
50	7.3	21	24.2	29	71.2
75	9	25	20.4	24.1	49.4
100	10.5	29	18.1	21.2	38.4
150	13.1	34	15.5	17.7	27.5
200	15.5	39	13.9	15.8	22
250	17.7	42	12.9	14.4	18.8
300	19.8	46	12.1	13.5	16.6
350	21.8	48	11.5	12.7	15
400	23.7	51	11.1	12.1	13.8
450	25.5	53	10.7	11.7	12.9
500	27.4	55	10.4	11.3	12.2
550	29.2	57	10.1	10.9	11.6
600	31	58	9.9	10.6	11.1

[a] Including the traffic operation costs, the waiting time costs, and the costs of riding time when the bus stops for boarding/alighting.

CONTRASTS WITH CURRENT PRACTICE

From the preceding account it is clear that a bus service designed to minimise the total social costs will look radically different from most existing bus services, principally because (i) the frequency of service will be higher in general, and particularly in off-peak, and (ii) the bus size will be much smaller in general, and particularly on thin routes.

It is interesting to examine the magnitude of the social cost savings that these differences in design can yield. The cost comparison above includes three alternatives. On the one hand, we consider these two variants of total social cost minimisation:

1. The case just dealt with, where the bus size is a control variable as well as the number of buses. For computational reason the "minority case" where $N_0 = N_1$ is considered. It is manifestly not generally optimal to equalise the frequency of service when the bus size is variable. However, the overestimation of the minimum social costs that can result is quite insignificant.
2. The case dealt with before, where the bus size is given, and the number of buses is the sole control variable.

It is not easy to choose an alternative for the comparison which is representative of current practices without tending towards caricature. A simple and clear objective has to be assumed, however, in order to carry out the calculations. This alternative representing "current practices" will be considered:

3. The bus size is given, and the capacity—number of buses—is adjusted so that the occupancy rate will equal ϕ_{max} both in peak and in off-peak.

The total social costs per passenger are calculated for each of these three cases.

Setting $N_0 = N_1$, and inserting the values for F^{opt} and S^{opt} found in the preceding section in the total social cost expression (21), gives the total social cost per day in optimum. Dividing this by EB gives the social cost per passenger in optimum.

$$AC^{opt} = \frac{at}{\beta E} + \frac{bhJ}{E\phi_{max}} + chJ + 2\sqrt{\frac{ahJ\left(\frac{v}{2} + \left(c\beta + \frac{b}{E\phi_{max}}\right)tQ_1\right)}{\beta EQ_1}} \tag{35}$$

The third and fourth terms are very dominating. The third term stands for the passenger cost of riding time proper. Given the running speed, it is clear that this cost is proportional to the journey length J. The fourth term is a conglomerate of passenger costs of waiting time and stop time (on the bus) and bus company costs.

In the second case considered, the total social cost in optimum is obtained by inserting expression (16a) for the optimal frequency (given the bus size) in the total social cost expression (12). Dividing this by the total number of passengers per day yields this cost per passenger.

$$AC^{opt}_{\text{given } S} = \frac{tIC}{\beta E} + chJ + 2\sqrt{\frac{hJ\left(\frac{v}{2} + ctQ_1\right)IC}{\beta EQ_1}} \tag{36}$$

In the third case it is assumed that the bus size is given, and that the number of buses is adjustable so that the occupancy rate ϕ_{max} applies both in peak and in off-peak. Under these assumptions the cost per passenger takes this shape:

$$AC^{\phi=\phi_{max}}_{\text{given } S} = \left(t + \frac{hJ}{\phi_{max} S}\right)\frac{\alpha IC_0 + IC_1}{\beta E} + chJ + ct\phi_{max} S + \frac{v\phi_{max} S}{2\beta Q_1} \tag{37}$$

As is seen from the three average cost expressions, chJ is a common tern to all: that is to say, the cost of riding time proper is by assumption independent of the design of the bus service. In the cost comparison in Table 7 this term is left out of consideration. What is compared is consequently the sum of the bus company costs and the costs of passenger waiting time and boarding/alighting time.

The same parameter values as have been previously used have been inserted in (35), (36), and (37). In the last two cases, where the bus size is given, the value of S is set = 75.

As expected, the cost difference is very striking in the low range of demand. The most important improvement on "current practices" that can be made is apparently to increase the frequency of service, especially in off-peak. But in addition the difference between $AC^{opt}_{S=75}$ and AC^{opt} is appreciable for low to moderate flows. Optimising the bus size makes a non-negligible contribution to the total improvement that can be made on "current practices".

When only the bus company costs are considered, the cost picture looks radically different. Under the assumptions made in the third case, the bus company cost is

constant throughout. The cost level represented by $PAC_{S=75}$ will be below both the other two producer costs in the whole interesting range of demand. It is evident that an optimal bus service will be very costly in a *narrow* sense, i.e. when only the bus company costs are considered. The change from the current position to an optimum would reduce the bus user costs very considerably, but at the expense of an appreciable increase in the costs of the bus company.

This no doubt poses a problem in view of the very strained financial situation of most urban bus companies. Who dares to make the costly quality improvements which are indicated, and raise bus fares to the required extent? That this should be done in a case where a budget constraint has to be taken into account is the inescapable conclusion of the present analysis. A policy of improving quality and raising fares has in fact the potential to *improve* the financial situation for bus companies, given that the present service design is inefficient. A change implying the fulfilment of the efficiency condition would raise the total willingness to pay more than the total costs of the bus company.

A final reflection is that such daring on the part of the bus companies would be a very good thing for the state of the art of urban transport economics. It would put the time values used by transport economists to the acid test of transport service users' willingness to pay.

APPENDIX

One-man versus two-man operation

So far one-man operation has been assumed. It may seem "backward" even to raise the issue of OMO versus TMO. One of the main changes in urban bus operation in recent decades has been the change over to OMO from TMO, and in countries like Sweden it was completed many years ago. Is that an unquestionable advance from a social point of view? This question has no clearcut answer. There is one unknown factor involved, namely, the prerequisite for OMO that any elaborate fare differentiation has to be abandoned. Otherwise, times for boarding would become unreasonable. I do not believe, but it is at least possible, that, if they were known, the potential allocative benefits of fare differentiation would make a decisive difference for the OMO versus TMO issue. Still, in spite of the application of systems of more or less flat fares, the boarding time for OMO buses seems to be about two seconds longer than for TMO buses (see Cundill and Watts, 1973). An incidental question is whether this difference may warrant TMO instead of OMO under any circumstances.

This issue can be usefully addressed with the aid of the present model for social cost minimisation. It is surely more interesting to compare *optimal* OMO and TMO alternatives rather than to make a *ceteris paribus* comparison.

Let us first approach the issue in a more qualitative way to get a better understanding of the main factors involved. Later on, the model will be used in calculating "OMO- and TMO-regions" with respect to some critical factors.

The point of having a conductor as well as a driver is, of course, that it relieves the driver from fare-collecting duty so that all his time can be spent in driving; this will reduce the ratio of time at stops to running time, to the benefit of both the bus

TABLE 8
Bus Company (Producer) Costs per Passenger

Q_1	PAC^{opt}	$PAC^{opt}_{S=75}$	$PAC^{\phi=\phi max}_{S=75}$
25	17.9	20.6	5.6
50	13.5	15	5.6
75	11.6	12.5	5.6
100	10.5	11.1	5.6
150	9.2	9.4	5.6
200	8.5	8.4	5.6
250	7.9	7.7	5.6
300	7.6	7.2	5.6
350	7.3	6.9	5.6
400	7.1	6.6	5.6
450	7	6.3	5.6
500	6.8	6.1	5.6
550	6.7	6	5.6
600	6.5	5.8	5.6

company and the passengers. The question is whether this benefit is worth the heavy price of the conductor's wage cost?

It would be patently ridiculous to have a conductor aboard a long-distance (say, London to Glasgow) coach taking up fares while the bus is running. The majority of the passengers probably travel the whole route distance (London–Glasgow), and the whole fare-collection takes perhaps only a quarter of an hour of the conductor's time; that would be a very poor rate of utilisation of the conductor, assuming that he stays aboard during the whole journey. On the other hand, a conductor aboard the London–Glasgow train, which may consist of some ten coaches, is by no means a ridiculous proposition, because the conductor may in this case be busy on the whole journey with fare-collecting/controlling work.

The heart of the matter is apparently that two-man operation cannot be a real alternative unless a reasonable rate of utilisation of the conductor can be naturally achieved. Two equally important factors for the potential usefulness of a conductor are (i) the size of the whole vehicle, which determines the total occupancy, and (ii) the journey length per passenger, which determines the seat turnover. The ratio of these two factors gives the number of boarding passengers per vehicle-mile. Recalling that the occupancy rate ϕ is defined as the ratio of the passenger flow Q to the product of the bus size S and the service frequency F, and that Q equals the product of the total number of boarding passengers of a bus line per hour B and the ratio of the journey length to the route length J/R, we have:

$$\phi = \frac{Q}{SF} = \frac{BJ}{SFR} \tag{38}$$

Rearranging the factors shows that the ratio of the occupancy to the journey length equals the number of boarding passengers per bus mile.

$$\frac{\phi S}{J} = \frac{B}{FR} \tag{39}$$

If this number is large enough to keep the conductor reasonably busy, two-man operation can be viable. From the left-hand side of (39) it is seen that, as was pointed out in the previous example, two-man operation is out of the question on a long-distance coach where J takes a very high value. If we reduce J while keeping S (and ϕ) constant, it is obvious that a situation will sooner or later be reached where the number of passengers boarding per bus-mile will be large enough to keep a conductor constantly busy. This is the most favourable situation for TMO versus OMO. (Which, however, does not necessarily mean that TMO is preferable from a social point of view. This is a further matter of relative factor prices.) In certain cases in urban traffic the mean journey length can be so short that, provided quite large buses are employed, a very high rate of utilisation of a conductor could be obtained.

The main point to be made in this connection on the basis of the present bus line model, where the bus size is a control variable, is that S is strongly dependent on J. Given the passenger flow, a 10% decrease in J will result in a 5% reduction in S^{opt} according to the finding that S^{opt} is proportional to the square root of J. This means that the combination of circumstances in which a high rate of utilisation of a possible conductor could be attained would be fairly rare in a case where the design of urban bus services is determined by social cost minimisation. It is quite conceivable that a conductor on a 75-seater double-decker serving markedly short-distance riders pays his way in terms of social cost savings, given this bus size, but it is also quite conceivable that buses of this size are far too large from a social point of view in most places where they are currently used. (For example, it is likely that halving the bus size and doubling the number of buses, and replacing TMO by OMO, would in many cases reduce the total social costs.)

With the aid of the model it is possible to specify under which conditions TMO can still be superior to OMO.

First, it can be interesting to note that assuming two-man operation in the model will raise the optimal bus size in the whole passenger flow range. By inserting TMO-values for the crew cost and the boarding/alighting time in the model, it is found that the optimal bus size is from 15% to 30% larger in the range from 50 to 1,000 peak passengers per hour, assuming two-man operation.

The next step is to calculate TC^{opt}/BE under TMO as well as OMO conditions and to find the difference. Using subscripts 1 and 2 for OMO and TMO, respectively, the cost difference per passenger takes this shape:

$$\frac{TC_1^{opt}}{BE} - \frac{TC_2^{opt}}{BE} = \frac{a_1 t_1 - a_2 t_2}{\beta E} + 2\sqrt{\frac{vhJ}{E}}\left(\sqrt{\frac{a_1}{20} + \frac{a_1 t_1 (c + b')}{v}} - \sqrt{\frac{a_2}{20} + \frac{a_2 t_2 (c + b')}{v}}\right) \tag{40}$$

where

$$b' = \frac{b}{\phi \beta E}$$

Consider first the two products a_1t_1 and a_2t_2. If these two products are equal, or if a_1t_1 is smaller than a_2t_2, TMO will never be viable, because both the first and second terms are then necessarily negative.

In other words: the whole issue would be settled from the start, if the percentage increase in the bus-size-independent traffic operation cost caused by adding a conductor were greater than the percentage decrease obtained in the boarding/alighting time. The parameter values assumed in the present study indicate, however, that $a_1t_1 > a_2t_2$. TMO can consequently not be ruled out without some further inquiry.

Inserting the previously used values for all the other parameters, it is found that the OMO–TMO cost difference is negative up to a peak flow level of at least 500 passengers per hour.

At this level social cost indifference rules between maintaining a service frequency of 30 OMO buses per hour, each holding about 50 passengers, and a service frequency of 24 TMO buses per hour, each holding 62 passengers.

The sensitivity of this result for changes in the parameter values assumed has been tested. It turns out that the most critical value is constituted by the ratio $(c + b')/v$. So far it has been assumed that $c = 50$, $b' = 13.5$, and $v = 150$ pence per hour. The value of waiting time has in several investigations been found to be at least three times as great as the value of riding time. It can be argued that this ratio is not quite relevant in the present connection, because the "riding time" involved constitutes in fact *stop time*, that is, passenger time consumed while the bus stands still for boarding/alighting. It is not unlikely that stop time is perceived as much more tedious than proper riding time, and that its value is close to the value of time spent waiting for the bus.

TABLE A1
OMO versus TMO: Critical Values of the Peak Flow

$\frac{c}{v}$ \ J	*1*	*2*	*3*	*5*	*10*	*15*
$\frac{1}{4}$	472	571	626	687	757	790
$\frac{1}{3}$	405	470	525	571	623	648
$\frac{1}{2}$	323	377	406	436	471	487
$\frac{3}{4}$	243	277	295	313	334	344
1	200	225	237	250	265	272

As is shown in Table A1, raising the value of c (that is, "c_{stop}" as distinct from "c_{ride}") to the level of v would tip the balance in favour of TMO in quite a substantial range. The figures in Table A1 represent the "critical values" of the peak flow, implying that at flow levels below a given value OMO is preferable, while TMO is preferable above the critical value. Not only the ratio c/v, but the mean journey length J, is varied. It is a bit surprising that this parameter did not turn out to be more important for the issue at stake. As has been mentioned, the reason is that the optimal bus size is strongly (positively) correlated with J.

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