MYOPIC INVESTMENT RULES AND TOLL CHARGES IN A TRANSPORT NETWORK

By S. C. Littlechild*

In a recent paper Dafermos and Sparrow [1] examined the problem of investment in a "user-optimised" transport network, in which travellers decide their paths on the basis of private costs only and ignore congestion costs imposed on others. Their model assumed that total traffic volume was given (i.e., completely inelastic), but they claimed that this was not a limitation.

> "The model developed in the paper assumes that demand is not affected by improvements . . . Total demand could be made a function of the average cost with no change in our major conclusions; all that would happen is that a demand elasticity term would appear in the calculations" (p. 185).

Their major conclusions were as follows:[1]

> "(a) If congestion tolls are ruled out as a means of burdening travellers with the costs they impose on others, then the determination of the investment pattern that minimises costs to users plus a capital rental charge on the investment itself is computationally infeasible.
>
> (b) If, on the other hand, we force individuals to explicitly consider the impact their travel has on others by means of congestion tolls (defined here as a charge to the traveller equal to the difference between the private (average) cost of the trip and the social (marginal) cost of the trip), then the choice of an optimal investment pattern as defined above becomes computationally feasible by standard marginal productivity methods.
>
> (d) In general, a system of tolls that charges the same amount to travellers on a link regardless of their destination cannot achieve the desired flow pattern that minimises total cost; rather, tolls must be charged on the basis of particular paths connecting origin-destination pairs. Thus, users must buy 'tickets' to the network, the cost of the ticket depending on the path chosen" (pp. 186–7).

Conclusion (a) is a rather pessimistic one to draw from the experience of formulating a problem which one cannot solve. Nevertheless, to be able to make investment decisions by means of "standard marginal productivity methods", that is, by "the myopic decision rule of sequentially choosing the best improvement" (p. 186), would be a decided advantage. Conclusion (b) is therefore "good news", but conclusion (d)

---

*University of Aston Management Centre, Birmingham.

[1]There was a further conclusion (*c*) which is difficult to understand.

seems to render totally impracticable a road pricing system, thereby losing not only the possibility of computing optimal investment plans, but also the possibility of optimally allocating traffic in a given network.

Conclusions (b) and (d) are not necessarily correct. Conclusion (b) generalises from a single form of the cost function; it is true in the particular case used by Dafermos and Sparrow, but to be generally true it requires the further assumption of diminishing returns to investment. Conclusion (d) is based on the incorrect assumption that equilibrium requires that "the travel times are equalised on *all* paths connecting a given origin to a given destination" (p. 185, italics supplied) rather than on all paths *actually used*.

We shall develop a model which seems simpler and more tractable than that of Dafermos and Sparrow, and which depends neither upon a specific form of cost function nor upon the assumption of fixed demand. We derive and discuss quite simple conditions under which a myopic investment rule is adequate for a single road, provided optimal congestion tolls are applied. There must be diminishing returns to investment, and either the road must be sufficiently congested or demand must be inelastic. The same conditions apply on a network of roads, where a link-by-link toll system suffices. These results indicate that, for all practical purposes, the introduction of congestion tolls would yield the benefits of myopic decision rules.

A SINGLE ROAD WITH FIXED CAPACITY

Following Walters [3], the diagram shows the marginal private cost curve or average social cost curve $g(y)$, the marginal social cost curve $c(y)$ and the demand curve $f(y)$ for a single road of fixed capacity. The two cost curves are related by two equivalent expressions for total cost:

$$\int_0^y c(x)dx = yg(y) \tag{1}$$

hence

$$c(y) = \frac{d}{dy}[yg(y)] \tag{1a}$$

$$= g(y) + yg'(y) \tag{1b}$$

$$= g(y)\,[1 + e(y)] \tag{1c}$$

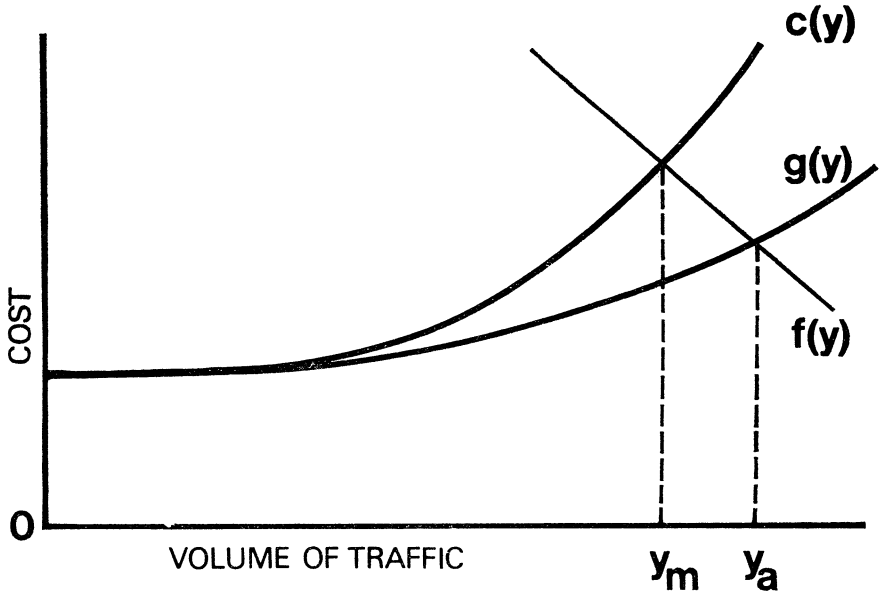
where $e(y)$ is the elasticity of the average cost curve.

Optimal usage on a "controlled" road is defined as that which maximises value of usage less total social cost:

$$\max_y \int_0^y [f(x) - c(x)]dx. \tag{2}$$

By differentiation we have at optimality

$$f(y) = c(y). \tag{3}$$



c (y) = marginal social cost
g (y) = average social cost (= marginal private cost)
f (y) = demand

At the optimal output, denoted y_m, marginal value of a trip equals *marginal* social cost.

Equilibrium usage on a "user-optimised" road can be thought of as determined by consumers maximising the value of output less *perceived* total cost:

$$\max_{y} \int_{0}^{y} [f(x) - g(x)]dx. \tag{4}$$

The optimality or equilibrium condition here is that

$$f(y) = g(y). \tag{5}$$

At the equilibrium output, denoted y_a, marginal value of a trip equals *average* social (or marginal private) cost.

The social optimum may be attained on a user-optimised road by the imposition of a toll equal to average cost multiplied by elasticity of average cost at the optimal usage volume. In this case consumers perceive the cost $[1 + e(y_m)]\ g(y)$ in (4). The optimality or equilibrium condition is therefore

$$f(y) = [1 + e(y_m)]\ g(y) = c(y) \qquad (5)$$

which is the same as (3).

We may therefore assume in the next section that a user-optimised road subject to an optimal toll has the same volume of traffic as a controlled road, and it will be more convenient to analyse the controlled road.

A SINGLE ROAD: INVESTMENT IN CAPACITY

Suppose that the capacity of the road can be increased by making an infinitely divisible investment costing s. Let the new cost curves be $g(y, s)$ and $c(y, s)$ and let the cost of capital be α. The problem is to choose levels of investment and usage so as to maximise value of usage less traffic cost and investment cost, or

$$\max_{y,\, s} V(y, s) \equiv \int_0^y f(x)dx - \int_0^y c(x, s)dx - \alpha s. \tag{6}$$

$$= \int_0^y f(x)dx - yg(y, s) - \alpha s.$$

Under what conditions is a myopic investment rule optimal?

A sufficient (but not necessary) condition is that V be a concave function, which requires that the Hessian matrix of second order partial derivatives

$$\begin{pmatrix} V_{yy} & V_{ys} \\ V_{ys} & V_{ss} \end{pmatrix}$$

be negative semi-definite. This in turn requires that

$$V_{yy} \leqslant 0 \tag{7a}$$

and

$$V_{ss} \leqslant 0 \tag{7b}$$

and

$$V_{yy}\, V_{ss} \geqslant V_{ys}^2. \tag{7c}$$

The various derivatives are as follows (omitting arguments):

$$V_y = f - g - yg_y \tag{8a}$$

$$V_{yy} = f' - 2gy - yg_{yy} \tag{8b}$$

$$V_s = -yg_s - \alpha \tag{8c}$$

$$V_{ss} = -yg_{ss} \tag{8d}$$

$$V_{ys} = -g_s - yg_{sy}. \tag{8e}$$

We shall assume that the demand function is not upward-sloping, that average cost is non-negative and that traffic and investment are strictly positive:

$$f' \leqslant 0,\ g(y, s) \leqslant 0,\ y > 0,\ s > 0. \tag{9}$$

We also make the following mild assumptions on the cost curve.

For given investment in capacity, average cost is a non-decreasing function of traffic volume:

$$g_y \geqslant 0. \tag{9a}$$

For given capacity, average cost as a function of traffic volume increases at a non-decreasing rate:

$$g_{yy} \geqslant 0. \tag{9b}$$

For given traffic volume, average cost is a non-increasing function of investment in capacity:

$$g_s \leqslant 0. \tag{9c}$$

For given traffic volume, average cost as a function of investment decreases at a non-increasing rate (non-increasing returns to investment):

$$g_{ss} \geqslant 0. \tag{9d}$$

The rate of increase of average cost with respect to traffic volume is a decreasing function of investment in capacity. Similarly, the rate of decrease of average cost with respect to investment is an increasing function of traffic volume:

$$g_{ys} = g_{sy} \leqslant 0. \tag{9e}$$

These assumptions immediately imply that inequalities (7a) and (7b) hold, since

$$V_{yy} = f' - 2g_y - yg_{yy} \leqslant 0 \tag{10a}$$

by (9), (9a) and (9b) and

$$V_{ss} = -yg_{ss} \leqslant 0 \tag{10b}$$

by (9d). In order that V may be concave, it is further required that (7c) hold, which may be rewritten

$$yg_{ss}(2g_y + yg_{yy} - f') \geqslant (g_s + yg_{sy})^2. \tag{10c}$$

Evidently a myopic investment policy for a single road is the more likely to be optimal:

(a) the higher is the rate at which average cost increases with volume of traffic (g_y), the more quickly that rate increases (g_{yy}), and the more steeply returns to investment decrease (g_{ss});
(b) the more inelastic is the demand; and
(c) the lower is the rate at which average cost decreases with investment (g_s) and the more slowly that rate increases with traffic (g_{sy}).

Intuitively, a myopic investment rule is most likely to be applicable on congested roads, where investment has rapidly diminishing returns and demand is inelastic. If demand is completely inelastic ($f' = -\infty$) at non-zero traffic level ($y > 0$), a myopic rule is applicable provided there are diminishing returns to scale ($g_{ss} > 0$).

On the other hand, these conditions for a myopic policy do not hold if investment does reduce average cost ($g_s < 0$) and if either

(a) there is no congestion ($g_y = g_{yy} = 0$) and demand is completely elastic ($f' = 0$), or
(b) there are constant returns to investment ($g_{ss} = 0$).

TWO APPLICATIONS

These conditions may be more easily appreciated in the context of particular examples. Consider first the cost function used by Dafermos and Sparrow, namely (in our notation):

$$g(y, s) = \frac{\bar{g}y}{\bar{s} + s} + d, \qquad \bar{g}, \bar{s}, d \geqslant 0. \tag{11}$$

It may be verified that assumptions (9a) to (9e) all hold; hence conditions (10a) and (10b) are satisfied. A myopic rule is appropriate if (10c) is also satisfied. In the present case this becomes

$$y \frac{2\bar{g}y}{(\bar{s}+s)^3} \left(\frac{2\bar{g}}{\bar{s}+s} + 0.y - f' \right) \geqslant \left(\frac{-\bar{g}y}{(\bar{s}+s)^2} - \frac{-\bar{g}y}{(\bar{s}+s)^2} \right) \tag{12}$$

which reduces to $f' \leqslant 0$, which is simply the requirement in (9) that the demand curve must not slope upwards. Hence, with the form of cost function (11) a myopic decision rule is always appropriate.

As a second example, consider an average cost function of Cobb-Douglas form:

$$g(y, s) = Ay^{\beta}s^{\gamma} \tag{13}$$

where we assume that elasticity of average cost is positive with respect to traffic volume ($\beta \geqslant 0$) and negative with respect to investment ($\gamma \leqslant 0$). It may be verified that assumptions (9a) and (9c)–(9e) hold in this case. Assumption (9b) holds if we further assume that average cost is elastic with respect to traffic volume ($\beta \geqslant 1$). The original assumptions are sufficient for conditions (10a) and (10b) to hold, for

$$V_{yy} = f' - \beta(1+\beta)\frac{g}{y} \leqslant 0$$

and

$$V_{ss} = -y\gamma(\gamma-1)\frac{g}{s^2} \leqslant 0.$$

Condition (10c) may be expressed as

$$\gamma(\gamma-1)\frac{yg}{s^2}\left[2\beta\frac{g}{y} + \beta(\beta-1)\frac{g}{y} - f'\right] \geqslant \left(\frac{\gamma g}{s} + \frac{\beta\gamma g}{s}\right)^2$$

which reduces to

$$g(1+\beta)(\beta+\gamma) \geqslant (1-\gamma)yf'. \tag{14}$$

A sufficient condition on the (Cobb-Douglas) cost function alone may be obtained by assuming infinite elasticity of demand. Putting $f' = 0$ in (14) yields

$$\beta+\gamma \geqslant 0. \tag{15}$$

If cost is more elastic with respect to traffic than with respect to investment (in absolute terms), a myopic rule suffices. If average cost is assumed elastic with respect to traffic ($\beta < 1$), it is sufficient for it to be inelastic with respect to investment ($|\gamma| < 1$).

On the other hand, if (15) does not hold we may rewrite (14) as

$$|e_d| = -\frac{(1+\beta)g}{f'y} \leqslant \frac{1-\gamma}{|\beta+\gamma|} \tag{14'}$$

where e_d denotes elasticity of demand.[2]

Since $\beta \geqslant 0$, we have

$$\frac{1-\gamma}{|\beta+\gamma|} = \frac{1-\gamma}{-\beta-\gamma} \geqslant \frac{1-\gamma}{-\gamma} = 1 - \frac{1}{\gamma} > 1$$

Hence a sufficient condition for a myopic decision rule is that demand be inelastic, or $|e_d| \leqslant 1$.

Are these conditions likely to be satisfied in practice? Walters [3] refers to empirical

---

[2]Defined by

$$e_d = \frac{f}{f'y} \leqslant 0.$$

From the optimality condition (3) f is given by

$$f = c(y) = g + yg_y = (1+\beta)g.$$

work suggesting the following values of elasticity of cost with respect to traffic: for uncongested roads, β of the order of 0·2 (his Table 2); for a particular congested road in Charleston, West Virginia, β around unity for traffic in the range 5 to 9 vehicles per minute, but increasing sharply to infinity as traffic increases to 10 vehicles per minute (his Figure 4). As regards elasticity of cost with respect to investment, Walters ([4], page 181) presents Venezuelan data showing that a gravel road requires about 180 per cent higher annual investment than an earth road but yields a decrease of 22 per cent in operating costs; that is an elasticity of $\gamma = -0{\cdot}12$. Similarly a bituminous road costs between 180 per cent and 280 per cent more than a gravel road and yields a decrease of 53 per cent in operating costs, which is an elasticity of between $\gamma = -0{\cdot}3$ and $\gamma = -0{\cdot}19$. For urban road he suggests that diminishing returns apply, which implies a lower (absolute) value for γ.

Even in the absence of information on elasticity of demand we may conclude, using the Cobb-Douglas model (15) as a basis, that the myopic rule will hold (i) for congested urban roads, and (ii) for uncongested urban roads provided that the diminishing returns conjecture holds. For rural roads, particularly in developing countries, we may calculate a limit on demand elasticity. Take traffic elasticity equal to half the lowest figure above, i.e. $\beta = 0{\cdot}1$, and take investment elasticity equal to twice the highest figure above, i.e. $\gamma = -0{\cdot}6$. Then from equation (14′) absolute demand elasticity would need to exceed 3·2 before a myopic rule ceased to be applicable. Such a high demand elasticity seems extremely unlikely. We may fairly conclude that a myopic decision rule will be generally applicable for road investments.

A ROAD NETWORK

When considering the imposition of tolls in a network of roads, Dafermos and Sparrow [1] seem to have been misled by their assumption that in equilibrium "the travel times are equalised on *all* paths connecting a given origin to a given destination" (p. 185, italics supplied). This leads them to error in the formulation of their equation (36) and in the attempt to solve it.

The impossibility of equalising travel times on all paths is evident from a simple three-node network. If travel times are equal for both paths from A to C (i.e. path AC and path ABC), they cannot be equal for both paths from B to C (i.e., BC and BAC).

Rather, equilibrium requires that travel times are equal for all paths *actually used* between a given origin and destination, and less than or equal to travel times on paths not used. One cannot in general specify which paths will be used without solving the allocation problem.[3]

In order to do this, one may formulate a programming model for a network which

[3]Marchand [2], investigating the optimal toll on one road when a substitute road cannot be taxed, also assumes, in his equation (4) on p. 576, that both roads are utilised so that average costs, including the toll in the one case, will be equal at equilibrium. It is difficult to see how the technique used could be generalised to cover a network of more than two nodes.

is the analogue of the single road model used so far. The model chooses volumes of trips between ultimate origins and destinations and routes them along intermediate links so as to maximise total value of trips less total cost on links. This programming model has been relegated to an Appendix because the notation is at first sight quite complex. But the equilibrium condition of the model can be easily stated. For each origin and destination pair, the marginal value of a trip is equal to the sum of the marginal costs on the cheapest route or routes; routes with higher costs are not used. For a "controlled" network the marginal costs referred to are social costs; for a "user optimised" network they are private costs. What is immediately apparent from the formulation is that a user-optimised network can be led to optimality by a toll imposed on each link, calculated in the way described earlier for a single road. The total toll paid by any traveller therefore consists of the sum of the tolls on the route he chooses. For each route used by travellers between a specified origin and destination pair, the total cost including time and tolls will be the same, and less than on any route not used by them.

The final question is whether myopic decision rules continue to apply in a network. Essentially, the answer is yes. In the Appendix we consider the case of arbitrary demand functions and show that a myopic decision rule is applicable to a network if it is applicable to each link of that network taken separately.

CONCLUSIONS

We have considered whether investment decisions can be made on a myopic basis on a single road and in a road network, provided an optimal system of congestion tolls is imposed. We have analysed the conditions on the demand and cost functions under which myopic rules are appropriate. Our general conclusion is that, in practice, they will normally be appropriate.

The usefulness of this analysis depends upon the uncertain possibility of imposing optimal congestion tolls. It is therefore interesting to consider investment in a user-controlled network without congestion tolls. It is perhaps worth indicating where the difficulty of analysis lies. Taking a single road for simplicity, users maximise

$$\int_0^y f(x)\,dx - \int_0^y g(x, s)\,dx \tag{16}$$

with respect to traffic volume y for given investment s. The road authority maximises

$$\int_0^y f(x)\,dx - \int_0^y c(x, s)\,dx - \alpha s \tag{17}$$

with respect to investment s, knowing that traffic volume y will be chosen by the above criterion. If the users' problem (16) could be solved for y as a function of s this function could be plugged into (17) and solved for s. This is the approach which Dafermos and Sparrow took. Unfortunately, the functional dependence of y on s from (16) is in general not easily characterised explicitly, especially in a network. The present analysis has been possible because in a controlled network both investment and traffic volume may be determined for the same purpose.

APPENDIX

The previous model extends quite naturally to a network of roads. Let $f^{ij}(x)$ be the demand function for travel from origin i to destination j, let x^{ij} be the number of such trips made, let y^{ij}_{kl} be the number of such trips routed between nodes k and l, and let $c_{kl}(y)$ and $g_{kl}(y)$ be, respectively, the marginal and average social cost functions on link (k, l). Optimal usage in a controlled network is that set of x^{ij} and y^{ij}_{kl} which maximises value of travel between all origins and destinations less social costs on all links:

$$\max \sum_{ij} \{\int_0^{x^{ij}} f^{ij}(x)\, dx - \sum_{k,l} \int_0^{\sum_{ij} y^{ij}_{kl}} c_{kl}(y)\, dy\} \tag{18}$$

subject to the network flow conservation constraints that for each node l the inflow corresponding to each origin and destination pair (i, j) must equal the outflow:

$$\delta^i_l(x^{ij}) + \sum_k y^{ij}_{kl} = \sum_m y^{ij}_{lm} + \delta^j_l(x^{ij}) \tag{19}$$

where $\delta^i_l(x) = x$ if $l = i$ and is zero otherwise. (x^{ij} appears in the constraint where node l happens to be the origin i and/or the destination j.)

The constraints (19) are linear and the objective function (18) is concave if the demand functions are non-increasing and the cost functions non-decreasing. In this case the Kuhn–Tucker conditions are necessary and sufficient for optimality. Let u^{ij}_l be the typical dual variable. For each origin and destination pair (i, j) we have

$$f^{ij}(x^{ij}) \leqslant u^{ij}_i - u^{ij}_j \tag{20}$$

with equality if $x^{ij} > 0$ at optimality. For all links (k, l) and for all origin and destination pairs (i, j) we have

$$u^{ij}_l - u^{ij}_k \leqslant c_{kl}(\sum_{ij} y^{ij}_{kl}) \tag{21}$$

with equality if $y^{ij}_{kl} > 0$ at optimality. Upon substitution

$$f^{ij}(x^{ij}) \leqslant c_{ik}(\sum_{ij} y^{ij}_{ik}) + c_{kl}(\sum_{ij} y^{ij}_{kl}) + \ldots + c_{mj}(\sum_{ij} y^{ij}_{mj}) \tag{22}$$

for all routes $i, k, l, \ldots, m, j$ with equality holding for those routes which are actually used. Equation (22) is the equivalent of (3) for one single road. It says that the optimal number of trips from origin i to destination j is such that the marginal value of an additional trip is equal to the sum of the marginal social costs on the cheapest route or routes. A route with higher costs is not used by travellers from i to j.

By replacing the marginal social cost functions $c_{kl}(y)$ by the marginal private cost functions $g_{kl}(y)$ one can generate the analogous characterisation of equilibrium in a user controlled network, namely

$$f^{ij}(x^{ij}) \leqslant g_{ik}(\sum_{ij} y^{ij}_{ik}) + g_{kl}(\sum_{ij} y^{ij}_{kl}) + \ldots + g_{mj}(\sum_{ij} y^{ij}_{mj}) \tag{23}$$

with equality holding for those routes which are actually used. Equation (23) corresponds to (5) for the single model; the interpretation is as for (22), replacing the word "social" by "private".

Now evidently on each link a toll may be set, equal to average cost multiplied by cost elasticity (both evaluated at socially optimum usage). The user controlled network is thereby led to social optimality.

In order to examine the applicability of myopic decision rules in the network, let the cost functions c_{kl} ( ) and g_{kl} ( ) now be written as functions also of investment s_{kl} on each link. For simplicity we shall confine ourselves to the case of arbitrary demand functions (i.e. $f'_{ij} = 0$). A sufficient condition for concavity of the objective function is that each term be concave; hence the total cost function on each link (k, l)

$$C_{kl}(\ldots, y_{kl}^{ij}, \ldots; s_{kl}) = \int_0^{\sum_{ij} y_{kl}^{ij}} c_{kl}(y, s_{kl})\, dy$$

$$= \sum_{ij} y_{kl}^{ij} g_{kl}\left(\sum_{ij} y_{kl}^{ij}, s_{kl}\right)$$

must be convex. (This approach is tantamount to calculating sufficiency conditions for demand functions of arbitrary elasticity.)

Let us simplify notation by omitting link subscripts and replacing the double origin and destination superscripts by a single subscript. We therefore wish to find conditions under which

$$C(y_1, \ldots, y_n; s) \equiv \int_0^{\Sigma y} c(x, s)\, dx = (\Sigma y)\, g\,(\Sigma y, s) \tag{24}$$

is convex. Denoting $\partial C/\partial y_1$ by C_1, etc., the condition is that the Hessian matrix

$$H = \begin{pmatrix} C_{11} & \ldots & C_{1n} & C_{1s} \\ \ldots & \ldots & \ldots & \ldots \\ C_{n1} & \ldots & C_{nn} & C_{ns} \\ C_{s1} & \ldots & C_{sn} & C_{ss} \end{pmatrix}$$

must be positive semi-definite. But notice that any derivative with respect to (say) y_i is the same as that with respect to y_j, for all i, j. Thus the Hessian matrix reduces to

$$H = \begin{pmatrix} C_{yy} & \ldots & C_{yy} & C_{ys} \\ \ldots & \ldots & \ldots & \ldots \\ C_{yy} & \ldots & C_{yy} & C_{ys} \\ C_{ys} & \ldots & C_{ys} & C_{ss} \end{pmatrix}$$

It is easily verified that any sub-determinant containing more than two rows and columns is zero, so that the conditions for H to be positive semi-definite are simply

$$C_{yy} \geqslant 0,\ C_{ss} \geqslant 0,\ C_{yy}\, C_{ss} \geqslant C_{ys}^2\,.$$

In terms of the average cost function, these are

$$2g_y + (\Sigma y)\, g_{yy} \geqslant 0, \tag{25a}$$

$$(\Sigma y)\, g_{ss} \geqslant 0 \tag{25b}$$

and

$$[2g_y + (\Sigma y)\, g_{yy}]\, (\Sigma y)\, g_{ss} \geqslant [g_s + (\Sigma y)\, g_{sy}]^2\,. \tag{25c}$$

These conditions are precisely analogous to (10a), (10b) and (10c) evaluated for $f' = 0$.

Thus, a myopic decision rule is applicable to a network of roads with arbitrary demand functions if it is applicable to each link of that network taken separately.

REFERENCES

[1] Dafermos, Stella, and F. T. Sparrow: "Optimal Resource Allocation and Toll Patterns in User Optimised Transport Networks." *Journal of Transport Economics and Policy*, Vol. V, No. 2, May 1971, pp. 184–200.

[2] Marchand, M.: "A Note on Optimal Tolls in an Imperfect Environment." *Econometrica*, Vol. 36, No. 3–4, July–October 1968, pp. 575–581.

[3] Walters, A. A.: "The Theory and Measurement of Private and Social Cost of Highway Congestion." *Econometrica*, Vol. 29, 1961, pp. 676–99.

[4] Walters, A. A.: *The Economics of Road User Charges*. Occasional Paper No. 5. International Bank for Reconstruction and Development, 1968.