DEMAND AND CHOICE MODELS OF MODAL SPLIT

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Recent investigations of modal choice have concentrated on the analysis of existing situations. Of particular interest has been the estimation of various elasticity measures and of the average value of time.

All models have employed the *trip* as the variable of interest. Two mechanically distinct approaches have emerged. One is to use *aggregates* of trips between two zones as the quantity of trips demanded. This entity is specified as a function of the out-of-pocket costs and travel times by each mode and some average zonal values of some socio-economic variables. The other approach uses individual trip data, so that the quantity is always zero or unity for a mode, according to whether or not it is chosen. This model is *binary choice* in the two-mode case; the choice depends on the individual values of times and costs by mode and on the characteristics of people who take the trips.

In the appendix we provide the theory necessary to produce, in the binary choice case, a utility interpretation of the estimated function, and in the aggregate case a demand interpretation of the function. The main result of the binary case is that the indirect utility function (a function of out-of-pocket costs, travel times, car expenditures, expenditures on housing, and income) can be estimated from empirical data, and this enables us to test hypotheses about the amounts and directions of changes in the variables. For the aggregate model we obtain for each mode demand equations which contain both money prices and travel times for all modes.

RECENT EMPIRICAL WORK

Quandt and Baumol

Several recent works[1] on both intercity and urban travel have specified models of the *aggregate* type, which are characterised as "demand" models. These models are analogous to demand, since they employ quantities of trips between zone or city pairs as dependent variables and include travel times and costs and income as independent variables. They have been designed to replace traditional methods, described in [23], [24], [25] and [26].

The model presented by Quandt and Baumol in, say, [16] is from the econometric standpoint a special case of the one presented by Wohl and Kraft in, say, [5].

[1]See [3], [5], [11], [16], [17], and [20]. [20] is a study made by Wohl and Kraft on intercity travel in the Northeast Corridor. [3], [5] and [11] are reports of later work done by them on urban travel data collected for Boston and applied to San Francisco. [16] and [17] are the results of some work by Quandt and Baumol on the Northeast Corridor project using intercity travel data collected in California.

Quandt and Baumol's "abstract mode" model is a single equation intercity demand model with travel by different modes between two given cities constituting distinct observations, but appearing in the same equation. This contrasts with the model specified by Wohl and Kraft, which employs a distinct demand equation for each mode. If the independent variables in both models are the same, the abstract mode model is a special case of the multi-mode model where the coefficient of each variable, in an equation which has travel by a given mode as the dependent variable, is equal to the corresponding coefficient in every other equation.[2] Another way to state the assumption is to say that the cross-elasticity of demand for travel by a mode is zero with respect to a change in travel time or cost by another mode which is not the best one, and with respect to the best one is constant.

Except for the difficulty mentioned above, the abstract mode model is an interesting exercise in economic model building. The idea of Quandt and Baumol that trips can, in some sense, be represented by a perfect substitutes model has emerged elsewhere. Further, the model employs, as exogenous variables, the ratio of the travel time of the given mode to that of the best mode, and the absolute travel time by the best mode. The authors show, for a simplified Cobb-Douglas version of the model, that if a new and faster mode is introduced on a route, and if at the same time total travel increases, the elasticity of travel with respect to relative travel time exceeds, in absolute value, the elasticity of travel with respect to the best absolute time. This is an intuitively plausible but not obvious restriction, which can be used in model building or tested for. The authors tested the hypothesis and found that it held for their data.

Wohl and Kraft

The most recent empirical work of Wohl and Kraft [3] and [5] employed quantities of trips by mode as dependent variables. They used an extensive set of independent variables, such as in- and out-of-vehicle travel times for both modes and linehaul and out-of-pocket costs for both modes, as well as some variables measuring aspects of population, employment, income and car ownership.[3] These variables were entered in both linear and natural log forms in all equations. The authors were unwilling to use ordinary least squares regression, since they expected some of their estimated elasticities to have the "wrong" sign and they felt that multicollinearity might emerge from the inclusion of so many variables. They used an estimation procedure which allows inequality constraints on the parameters to be built into the model.[4] A difficulty with the constrained estimation procedure is that the standard tests of ordinary least squares which are derived from normal sampling assumptions are no longer applicable. But in [3], pp. 109–111, it is stated that the computed R^2 is a lower bound of the true R^2, which seems reasonable. Another difficulty is the lack of standard errors or t-statistics on the estimates of the individual coefficients. Their results were interesting but not spectacular. Considering the inordinate number of variables included, the values of R^2 were uniformly small, ranging from about 0.35 to 0.65 for

[2]Chow's test could be applied to the more general model to determine whether or not this is a reasonable hypothesis. See [4] or [10].

[3]Not all the socioeconomic variables were entered in any single equation.

[4]See [14] and [29].

work and shopping trips. The direct elasticities with respect to the times and costs had reasonable absolute and relative magnitudes, but the cross elasticities were smaller than one would expect, in most cases zero or nearly so. The direct elasticity of auto work trips for in-vehicle travel time was −0.82, but for out-of-vehicle time it was −1.437. That of auto work trips with respect to auto linehaul costs was −0.494, while for out-of-pocket costs it was −0.071. Turning to transit work trips, the direct elasticity with respect to transit line haul time was −0.39, but for transit excess time −0.709. The elasticity for transit linehaul cost was −0.09, and for transit excess time it was similarly −0.100. The only non-zero cross-elasticities of the eight possible were the elasticities of demand for auto work trips with respect to transit excess time (0.373) and with respect to transit linehaul costs (0.138).

Lisco

Several recent studies of modal choice have employed some version of the *binary choice* model.

Lisco [12] employed probit analysis, which fits a cumulative normal distribution to the data. The probability that a given choice will be made is the ordinate of the cumulative normal at the point Y, where Y is a "score" or linear combination of the values of the independent or continuous variables and where the parameters or unknowns in the linear combination are estimated by non-linear regression methods. Lisco did not attempt to classify observations by their probabilities or to calculate elasticities; he was primarily interested in estimating the value of commuters' time, which he found to be of the order of $2.50 per hour for a commuter whose average wage rate was about twice that figure. He also found the value this commuter placed on the comfort afforded by his own car to be about two dollars a day. Further, people were willing to pay an average of $7.20 an hour to avoid walking in the downtown area. This supports the widely held view that terminal travel time is more distasteful than in-vehicle time.

Thomas

Thomas [21] used logit analysis, which resembles probit in employing non-linear regression to estimate parameters. He classified observations with two versions of his model, which had a route choice (toll road *vs.* free road) rather than a mode choice. Both versions produced about 85 per cent correct choices. The difference between the two versions was in the way the data on travel time were collected; one version employed a test vehicle equipped with a measuring device, while the other version used the commuter's own estimates. The value of travel time, as in Lisco's study, was defined as the ratio of the estimated coefficients of difference in travel time and toll per person corrected for units, where, unlike Lisco, operating costs were not included in the cost difference variable. The model based on test vehicle data produced a value of time equal to $1.82 per hour, while the one based on reported data produced a value of $3.84 per hour.

Warner

Warner's study was primarily an analysis of the elasticity of choice. He employed two models on the same data, first using the discrimination-classification approach under the assumption that the group covariance matrices were the same but that the

prior probabilities were not. He then used the logit model, using the discrimination estimates for the first stage in the iterative process (see [8]), on the same data, and found that the two methods yielded similar results ([27] p. 75). Two approximations were made to estimate the elasticity of choice for car travel with respect to relative travel time and relative cost. Both results were that these two elasticities were about the same: they were of the order of 0.16 and 0.21, the differences being due to different ways of aggregating over individuals. The elasticity of choice for car travel with respect to income appeared to be quite small; so was that for a proxy variable for car ownership, but the latter was more important than the former. Neither variable was continuous and neither was a direct proxy of the variable in question, so no attempt was made to evaluate or interpret the values of elasticities.

Quarmby

Quarmby [18] investigated some Leeds (England) data on work trips, using a discrimination-classification approach. He used posteriors rather than elasticities to investigate sensitivities of the choice model to various hypothetical policy changes. He also calculated some values of travel time under several assumptions, finding that the value of commuter travel time is about 20 to 25 per cent of the commuter's wage rate and that this holds over a wide range of incomes. Finally, there was insufficient evidence to conclude that the value of time by bus was significantly different from that by car.

Beesley

Beesley [2] studied London data on work trips, and found that time is valued at about one-third the wage rate for public transit riders earning about the average wage. For higher income workers the value of time increased to half the wage rate. It also appeared that time spent in car travel was valued higher than time spent in public transit.

RESULTS OF PRESENT STUDY

We studied San Francisco data collected by the Bay Area Transportation Study Commission, using a discrimination-classification model.[5] We considered, for the work trip model, the effects of several policies. In each case we considered the effect of a given relative change in one of the variables and traced the impact of that change on the percentage of traffic going by each mode.[6]

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[5]See [13]. The data from the O-D study were merged with network travel times and out-of-pocket costs.

[6]There is, of course, an effect on some and perhaps all the other variables from a sufficiently large change in any of the variables. For example, if relative travel cost is decreased more people ride transit, causing the road to be less congested, so that relative travel time increases, causing more people to take cars. There is no simple way out of this problem. One could perhaps specify or estimate a dependence between variables and attempt to assess effects of a given policy, *e.g.*, a 30 per cent decrease in transit fares which concurrently implies a 5 per cent increase in relative travel time, where both these effects are assumed for computational purposes to occur simultaneously. Then effects could be worked out using both possible sequences and averaged.

The procedure involved a number of arbitrary decisions. We calculated the elasticity of auto choice with respect to each of the variables evaluated at the mean value of that variable for car travellers, and with the transit posterior likewise evaluated at all the mean values for the car travellers. Similarly, the elasticities of transit choice and their auto posterior component were calculated at the point of means for transit travellers. The effect of, for example, a 30 per cent increase in relative travel time, Vb/Va, was calculated as follows. Since the elasticity of auto choice with respect to relative travel time was estimated to be 0.500, the impact of a 30 per cent increase in relative travel time on the fraction of traffic going by automobile was assumed to be 15 per cent. The initial percentage of total traffic going by automobile was 78.6 per cent. Fifteen per cent of this is 11.79 per cent, which added to the initial percentage yields 90.39. But we must also consider the impact of the change on transit travel. The elasticity of transit choice with respect to relative travel time was 1.059. The impact of a thirty per cent decrease in the latter was 31.77, which, multiplied by 21.4, the initial percentage of trips by transit, yields 6.80. Subtracting the latter from the former yields 14.6. It does not appear reasonable to allow growth or decline in the model, because, for example, the 30 per cent increase in relative travel time could come from a 30 per cent increase in transit travel time, Vb, or from approximately a 23 per cent decrease in auto travel time, Va, since

$$1.3\frac{Vb}{Va} = \frac{1.3Vb}{Va} = \frac{Vb}{\frac{Va}{1.3}} = \frac{Vb}{.77Va}$$

Consequently, we do not wish to allow the fact that 90.39 added to 14.60 adds up to something other than 100. To adjust these figures we simply allow them to keep the same relative proportions but normalise them to sum to 100. Specifically, we calculate $100 \div 104.99 = 95.25$ and multiply this by 90.39 and 14.60 to obtain 86.10 and 13.91 for the adjusted automobile and bus proportions. Finally, subtracting the final bus proportion, 13.9, from the initial one, 21.4 ,we obtain the result that the decrease in bus travel due to a 30 per cent increase in relative travel time is 7.5 per cent.

The same procedure was applied to five different policies. The results appear in Table 1. (See the Appendix for definitions of the variables.)

TABLE 1

Policy	*Change in Bus Percentage*
30% increase in $\frac{Pb}{Pa}$	−5.6
(Free bus) 100% decrease in $\frac{Pb}{Pa}$	+22.3
30% increase in $\frac{Vb}{Va}$	−7.5
10% increase in $\frac{Vb}{Va}$	−2.6
10% decrease in C	+3.3

We can see in this table that the effects are roughly proportional to the elasticities of choice between variables and to the amount of change in a given variable. It is not obvious that there is a similar simple relationship between the initial elasticities of choice as applied to the initial percentages and the final change in the percentage going by each mode.

COMPARISON OF RESULTS

Quarmby's cost and time variables were defined as differences, not ratios: and, instead of using the elasticities of choice to calculate the changes, he calculated the changes in each individual posterior resulting from each policy change and averaged the posteriors to estimate the percentage change. His results, which were based on policies somewhat different but roughly similar to the ones we considered, are presented in part in Table 2.

TABLE 2

Policy	*Change in Bus Percentages*
(1) Bus fares increased by 6d.	−8
(2) Buses free	+17
(3) Cars 10 minutes faster	−9
(4) Buses 10 minutes faster	+9
(5) Parking cost increased by 1s.	+12

Our first policy is roughly comparable with Quarmby's and yields a somewhat smaller but similar number. The second policy is identical to his, but ours indicated a slightly stronger effect. Our third was similar to his third and fourth; again our effect was slightly smaller.

We obtained from Warner [27] some alternative estimates of elasticities of choice for auto and transit work trips with respect to relative times and cost. He estimated that these were approximately 0.2 and 1.0 for auto and transit respectively, and that time and cost ratios had about the same elasticities. Warner's estimates were inserted into two of the four policies, using our method and initial fractions. The results appear in Table 3.

Similarly, we obtained some estimates from Wohl and Kraft on Boston data where the model was estimated by constrained regression, the dependent variable was an aggregate of work trips between zone pairs with a different equation for each mode, and the independent variables included functions of absolute travel times and costs by both modes. Their elasticity estimates were also inserted into two of our policies and procedures to obtain alternative estimates of change in the proportion using transit. It was possible to calculate both policies two different ways, since both policies can be stated in different ways according to whether the numerator or denominator in the price ratio is assumed to change.

The results of Warner, Quarmby and Wohl-Kraft and mine are presented together in Table 3 for the policies of increasing relative travel time by 30 per cent and increasing relative travel cost by 30 per cent.

TABLE 3

Effect† on Bus Proportion

Policy	30% ↑*Vb*	23% ↓*Va*	30% ↑*Pb*	23% ↓*Pa*
Quarmby*	−9.0			−8.0
Wohl-Kraft	−7.3	−6.2	−1.6	−2.0
McGillivray		−7.5		−5.6
Warner		−6.4		−6.4

*See Table 2
†See text for explanation of estimates

In spite of considerable differences in methods and data base, we can note consistencies. The effect of changing travel cost is never stronger than that of changing travel time. My own estimates seem to lie midway between others. The elasticities of Wohl and Kraft produce about the same answer for both ways of calculating it. Quarmby's changes are higher than any others.[7] The travel time results contain little variability for such a crude comparison.

MATHEMATICAL APPENDIX

A Utility Model

To justify analysis of trip-making decisions separately from those on other commodities, we can simply assume that the cross-elasticities of demand for the commodities felt to be directly related to mode choice, *i.e.* travel, automobiles, and housing, with respect to the prices of other commodities are zero or nearly so. Alternatively, we can assume that consumers' choices for different commodity groups are considered separately. Specifically, we can assume that the decisions on transport and housing are separate from those on other commodities and are made in two stages. First the individual makes the long-run decision of choosing how much to spend on automobiles, urban travel, and residential space. Then he decides how to allocate this expenditure over these three commodities. His short-run decision on mode of travel is based on the situation confronting him with respect to the available automobile and transit services and their costs.

If we neglect all commodities outside the transport and housing sector, we can write the individual's utility function

$$U = U(R, C, S, T)$$

where U is ordinal utility and R, C, S, and T are quantities of urban travel, vehicle ownership or availability, residential space, and leisure time.

The budget constraint for this set of commodities is

$$Y_1 = P_a R + P_b(1-R) + P_c C + P_s S + P_t T$$

[7]This may be due to differences in definitions of policies.

where Y_1 is the amount of income allocated by the consumer unit to the commodity group containing travel and housing. P_a, P_b, P_c, P_s, and P_t are the prices of the commodities under consideration, the subscripts referring to commodities. The situation we wish to consider is that the decision is between auto and transit for a particular trip, so that we require R to take on the value one if auto is chosen and zero if transit is chosen. P_a and P_b are the out-of-pocket costs of an auto and transit trip respectively. P_t is the value of time spent in urban travel and is assumed to be equal for auto and transit.

We are interested in whether auto or transit will be chosen for any given value of the variables in U, *i.e.* for which values will

$$U_a(R, C, S, T) > U_b(R, C, S, T)$$

in which case auto will be chosen.

Substituting (2) into (3),

$$U_a\left(\frac{Y_1 - P_cC - P_sS - P_tT}{P_a}, C, S, T_a\right) > U_b\left(\frac{Y_1 - P_cC - P_sS - P_tT}{P_b}, C, S, T_b\right)$$

so that the decision is based on the expenditures on automobiles and housing, the value of time spent, the out-of-pocket costs of the auto and transit trips, the amounts of housing, car ownership, and time spent by the consumer, and the amount of expenditure he allocates to this commodity group. The form of the function U is, of course, unknown. We assume that the out-of-pocket costs of auto and transit enter into it in the form of the ratio of those for transit to those for auto. Similarly we assume that travel times enter into it as the ratio of transit travel time to auto travel time. Expenditure on housing is easier to measure than its components, price and quantity. Further, a physical unit of residential space, such as a square foot, may not mean the same thing for different types or qualities of housing. Therefore we employed expenditure as the housing decision variable. For similar reasons we employed the number of cars available to the household as the auto decision variable for mode choice. We assume, finally, that expenditure on the urban travel and housing commodity group is proportional to total income, so that we considered the latter instead of the former in the empirical form of the model. With these assumptions, the criterion becomes to choose auto if

$$U_a\left(\frac{P_b}{P_a}, \frac{T_b}{T_a}, C, P_sS, Y\right) > U_b\left(\frac{P_b}{P_a}, \frac{T_b}{T_a}, C, P_sS, Y\right) \quad .$$

In the empirical work described in the text we used the discrimination-classification model. Discrimination is a procedure for selecting an optimal hypersurface which divides the vector space defined by the set of variables into two regions. Classification is a procedure for deciding, for a new observation of unknown mode, which mode is more likely to be chosen.

The functional forms of the discriminant function are dictated by the assumptions on the populations of the underlying variables employed, *i.e.*, that the populations of auto and transit trip takers are each joint multivariate normal in the set of variables specified with known mean vectors and covariance matrices. Under the assumption that the two populations have different covariance matrices, the discriminant

function, and hence the indifference hypersurface of the constrained utility function, are quadratic in the variables. If we further assume that the covariance matrices of the two populations are the same, as is usually done in practical applications, the above-mentioned entities reduce to linear functions of the variables.

A Testable Hypothesis

If we assume that a car and a transit vehicle are available to the trip taker at the time the trip is desired, that trips are differentiated only[8] by their price and travel time characteristics[9] and that each trip taker's value of time is the same whether the trip is by auto or by transit, then we can infer an immediate hypothesis. The hypothesis is simply that if a mode both takes less time and costs less money it will be chosen. For example if $P_a < P_b$ and $T_a < T_b$ and the value of time is P_t, then $P_a + P_tT_a < P_b + P_tT_b$, *i.e.* auto is cheaper and is chosen for every trip for which this inequality holds.

A Demand Model for Urban Travel and Related Goods

Consider a consumer who has a utility function with several commodities as arguments,[10]

$$U^* = U^*(A, B, C, L, S, Z)$$

where U^* is ordinal utility, and A, B, C, L, S, Z are respectively quantities of the commodities auto travel, transit travel, automobiles, leisure, residential space, and a composite residual commodity.

Suppose the consumer has two stocks[11] of which he has fixed quantities and which he can spend on commodities

$$Y = P_a \cdot A + P_b \cdot B + P_c \cdot C + P_l \cdot L + P_s \cdot S + P_z \cdot Z$$
$$T = V_a \cdot A + V_b \cdot B + V_c \cdot C + V_l \cdot L + V_s \cdot S + V_z \cdot Z$$

while Y and T are money income and time, and P_i and V_i are unit prices in terms of money and time respectively for commodity i. Assume further that $V_c = V_s = 0$, *i.e.*, that the time prices of a unit of automobiles and a unit of housing both appear insignificant relative to the money prices.[12]

The constraints become

$$Y = P_a \cdot A + P_b \cdot B + P_c \cdot C + P_l \cdot L + P_s \cdot S + P_z \cdot Z$$
$$T = V_a \cdot A + V_b \cdot B + V_l \cdot L + V_z \cdot Z$$

Suppose the utility function is weakly separable[13] with respect to a partition of the

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[8]Flexibility, comfort and convenience are other differentiating characteristics.

[9]See [15] for discussions of commodities identifiable only by price characteristics.

[10]See [1] and [28] for theoretical supply and demand analyses of the choice between travel and residential space.

[11]See [6], [9], [19], [22] and [28]. In [28] it was observed that travel time could be viewed as a price in another currency so that the theory of rationing could be applied. In [9] this theory was applied to the choice between travel and leisure. [6] contains a simple diagrammatic analysis of consumer choice under rationing. [19] and [22] are more general mathematical theories of rationing.

[12]We mean here the time involved in the purchase of cars or housing. The time involved in the use would be charged to A, B or L.

[13]See [7] for a discussion of various kinds of independence between commodity groups.

commodities into two groups, I = (A, B, C, L, S) and II = $(\mathcal{Z})$, and that the allocation of money and time is likewise partitionable in advance of analysis. The model becomes

$$
\begin{aligned}
U &= U(U_1[A, B, C, L, S], U_z[\mathcal{Z}]) \\
\mathcal{Y}_1 &= P_a \cdot A + P_b \cdot B + P_c \cdot C + P_s \cdot S + P_l \cdot L \\
\mathcal{T}_1 &= V_a \cdot A + V_b \cdot B + V_l \cdot L \\
\mathcal{Y}_2 &= P_z \cdot \mathcal{Z} \\
\mathcal{T}_2 &= V_z \cdot \mathcal{Z} \\
\mathcal{Y} &= \mathcal{Y}_1 + \mathcal{Y}_2 \\
\mathcal{T} &= \mathcal{T}_1 + \mathcal{T}_2
\end{aligned}
$$

If both the utility function and the stocks of time and money are separable in the sense stated, the utility maximisation procedure leads to demand equations which are not dependent on the prices and stocks of the other group. Form the Lagrangian

$$
\begin{aligned}
\mathcal{L} = U &+ \lambda_1(\mathcal{Y}_1 - P_a \cdot A - P_b \cdot B - P_c \cdot C - P_s \cdot S - P_l \cdot L) \\
&- \lambda_2(\mathcal{T}_1 - V_a \cdot A - V_b \cdot B - V_l \cdot L) \\
&- \lambda_3(\mathcal{Y}_2 - P_z \cdot \mathcal{Z}) \\
&- \lambda_4(\mathcal{T}_2 - V_z \cdot \mathcal{Z})
\end{aligned}
$$

If we partially differentiate the Lagrangian with respect to each commodity and each Lagrange multiplier, we obtain

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A} &= \frac{\partial U_1}{\partial A}(\mathrm{A,B,}C,S,L) - \lambda_1 P_a - \lambda_2 V_a \\
\frac{\partial \mathcal{L}}{\partial B} &= \frac{\partial U_1}{\partial B}(A,B,C,S,L) - \lambda_1 P_b - \lambda_2 V_b \\
\frac{\partial \mathcal{L}}{\partial C} &= \frac{\partial U_1}{\partial C}(A,B,C,S,L) - \lambda_1 P_c \\
\frac{\partial \mathcal{L}}{\partial S} &= \frac{\partial U_1}{\partial S}(A,B,C,S,L) - \lambda_1 P_s \\
\frac{\partial \mathcal{L}}{\partial L} &= \frac{\partial U_1}{\partial L}(A,B,C,S,L) - \lambda_1 P_l \\
\frac{\partial \mathcal{L}}{\partial \mathcal{Z}} &= \frac{\partial U_z(\mathcal{Z})}{\partial \mathcal{Z}} - \lambda_3 P_2 - \lambda_4 V_2 \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= \mathcal{Y}_1 - P_a \cdot A - P_b \cdot B - P_c \cdot C - P_s \cdot S - P_l \cdot L \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= \mathcal{T}_1 - V_a \cdot A - V_b \cdot B - V_l \cdot L \\
\frac{\partial \mathcal{L}}{\partial \lambda_3} &= \mathcal{Y}_2 - P_z \cdot \mathcal{Z} \\
\frac{\partial \mathcal{L}}{\partial \lambda_4} &= \mathcal{T}_2 - V_z \cdot \mathcal{Z}
\end{aligned}
$$

If we set the partials of the Lagrangian equal to zero and solve the system in terms of the variables we obtain two groups of equations:

$$
\begin{aligned}
A &= A(P_a, V_a, P_b, V_b, P_c, P_s, P_l, V_l, Y_1, T_1)\\
B &= B(P_a, V_a, P_b, V_b, P_c, P_s, P_l, V_l, Y_1, T_1)\\
C &= C(P_a, V_a, P_b, V_b, P_c, P_s, P_l, V_l, Y_1, T_1)\\
S &= S(P_a, V_a, P_b, V_b, P_c, P_s, P_l, V_l, Y_1, T_1)\\
L &= L(P_a, V_a, P_b, V_b, P_c, P_s, P_l, V_l, Y_1, T_1)\\
\lambda_1 &= \lambda_1(P_a, V_a, P_b, V_b, P_c, P_s, P_l, V_l, Y_1, T_1)\\
\lambda_2 &= \lambda_2(P_a, V_a, P_b, V_b, P_c, P_s, P_l, V_l, Y_1, T_1)
\end{aligned}
$$

and

$$
\begin{aligned}
Z &= Z(P_z, V_z, Y_2, T_2)\\
\lambda_3 &= \lambda_3(P_z, V_z, Y_2, T_2)\\
\lambda_4 &= \lambda_4(P_z, V_z, Y_2, T_2)
\end{aligned}
$$

Under the usual regularity (convexity) conditions, the demand equations represent the reduced form of the utility maximisation problem.

Now let the normalisation conditions $P_l = V_l = 1$ hold. The system of demand equations reduces to

$$
\begin{aligned}
A &= A(P_a, V_a, P_b, V_b, P_c, P_s, Y_1, T_1)\\
B &= A(P_a, V_a, P_b, V_b, P_c, P_s, Y_1, T_1)\\
C &= C(P_a, V_a, P_b, V_b, P_c, P_s, Y_1, T_1)\\
S &= S(P_a, V_a, P_b, V_b, P_c, P_s, Y_1, T_1)\\
Z &= Z(P_z, V_z, Y_2, T_2)
\end{aligned}
$$

where we drop L, λ_1, λ_2, λ_3, λ_4 from the system, because the equations of interest are those remaining. The interesting equations are A and B, since these are demand equations for auto and transit which contain *both* prices and travel times. We proceed now to extend the harder theorems of consumer theory to the multiple constraint demand model.

The second-order conditions for the constrained maximisation of the utility function can be stated in terms of the (symmetric) bordered Hessian matrix

where
$$
\mathrm{H} = \begin{bmatrix}
U_{1ss} & U_{1sb} & U_{1sa} & 0 & P_s & 0 & 0 & 0\\
 & U_{1bb} & U_{1ba} & 0 & P_b & 0 & V_b & 0\\
 & & U_{1aa} & 0 & P_a & 0 & V_a & 0\\
 & & & U_{zz} & 0 & P_z & 0 & V_z\\
 & & & & 0 & 0 & 0 & 0\\
 & & & & & 0 & 0 & 0\\
 & & & & & & 0 & 0\\
 & & & & & & & 0
\end{bmatrix}
$$

$$
U_{1ba} = -\frac{\partial\left(\dfrac{\partial U}{\partial U_1}\ \dfrac{\partial U_1}{\partial b}\right)}{\partial a}, \text{ for example.}
$$

Owing to the block-triangularity of H and to the fact that we can interchange the fourth and fifth columns and rows and then the fifth and seventh columns and rows without changing the value of $|H|$, we can write $|H|$ as

$$|H| = |H^I| \cdot |H^{II}| \text{ where}$$

$$|H^I| = \begin{vmatrix} U_{1ss} & U_{1sb} & U_{1sa} & P_s & 0 \\ & U_{1bb} & U_{1ba} & P_b & V_b \\ & & U_{1aa} & P_a & V_a \\ & & & 0 & 0 \\ & & & & 0 \end{vmatrix}$$

and $|H^{II}|$ is the analogous determinant for the composite commodity.

The second-order conditions for constrained utility maximisation are that the southeast principal minors of $|H|$ of order larger than four have alternatively positive and negative signs, starting from negative.

Before introducing the testable theorems of demand analysis, we define the elasticity of a variable i with respect to another variable P_j

$$E_{iP_j} = \frac{P_j}{i} \frac{\partial i}{\partial P_j} = \frac{\frac{\Delta i}{i}}{\frac{\Delta P_j}{P_j}}$$

or, in words, E_{iP_j} is the relative change in i, given some relative change in P_j.

If we totally differentiate the first-order conditions for commodity group 1, we obtain

$$\begin{aligned}
U_{1ss}dS + U_{1sb}dB + U_{1sa}dA - P_s d\lambda_1 &= \lambda_1 dP_s \\
U_{1bs}dS + U_{1bb}dB + U_{1ba}dA - P_b d\lambda^1 - V_b d\lambda_3 &= \lambda_1 dP_b + \lambda_3 dV_b \\
U_{1as}dS + U_{1ab}dB + U_{1aa}dA - P_a d\lambda_1 - V_a d\lambda_3 &= \lambda_1 dP_a + \lambda_3 dV_a \\
-P_s dS - P_b dB - P_a dA &= -d\Upsilon_1 + SdP_s + BdP_b + AdP_a \\
- V_b dB - V_a dA &= -dT_1 + BdV_b + AdV_a
\end{aligned}$$

These are treated as a simultaneous system of five equations in the unknowns dS, dB, dA, $d\lambda_1$, $d\lambda_3$.

Using Cramer's rule to solve for, *e.g.*, dS, we find

$$dS = \frac{\lambda_1 dP_s \, |H_{ss}^I| + (\lambda_1 dP_b + \lambda_3 dV_b) \, |H^I{}_{bs}| + (\lambda_1 dP_a + \lambda_3 dV_a) \, |H_{as}^I|}{|H^I|} + \frac{(-d\Upsilon_1 + SdP_s + BdP_b + AdP_a) \, |H^I{}_{P_s}|}{|H^I|}$$

where, *e.g.*, $|H^I{}_{bs}|$ = cofactor of the element in the first column and the second row of H^I and $|H^I{}_{P_s}|$ = cofactor of the element in the first column and fourth row of H^I. Assuming $dP_b = dV_b = dP_a = d\Upsilon_1 = dT_1 = 0$, we find that

$$\frac{\partial S}{\partial P_s} = \frac{\lambda_1 \, |H_{ss}^I| + S \, |H^I{}_{P_s}|}{|H^I|}$$

Similarly

$$\frac{\partial B}{\partial P_b} = \frac{\lambda_1 |H_{bb}{}^I| + B |H^I{}_{pb}|}{|H^I|}$$

And, more generally,

$$\frac{\partial B}{\partial P_a} = \frac{\lambda_1 |H_{ba}{}^I| + A |H^I{}_{P_b}|}{|H^I|}$$

These last three equations have economic interpretations. In each, the first term is called the substitution effect and the second is called the income effect. For example:

$$\frac{\partial B}{\partial P_{a\bar{U}}} = \frac{\lambda_1 |H_{ba}{}^I|}{|H^I|}$$

where $\bar{U}$ denotes utility is being held constant. This is called the substitution effect of a change in auto cost on transit travel.

$$-A \frac{\partial B}{\partial Y_1} = \frac{A |H^I{}_{pb}|}{|H^I|}$$

is called the income effect of a change in auto cost on transit travel.

Two testable hypotheses can be derived:

1. $$\frac{\partial B}{\partial P_{b\bar{U}}} = \frac{\lambda_1 |H_{bb}{}^I|}{|H^I|} = \frac{\lambda_1 |H_{bb}{}^I| \, |H^{II}|}{|H^I| \, |H^{II}|} < 0$$

by second order conditions. This rule says that own-price substitution effects are negative.

2. $$\frac{\partial B}{\partial P_{a\bar{U}}} = \frac{\lambda_1 |H_{ba}{}^I|}{|H^I|} = \frac{\lambda_1 |H_{ab}{}^I|}{|H^I|} = \frac{\partial A}{\partial P_{b\bar{U}}}$$

by the symmetry of H. This rule says that the cross-price substitution effects are symmetric.

From Euler's Theorem and the fact that demand functions are homogeneous of degree zero in their arguments, we can deduce the following:

3. $$\sum_j E_{ij} = 0 \qquad i = S, B, A$$
$$j = P_a, P_b, P_s, V_a, V_b, Y_1, T_1$$

Finally, we introduce the following classifications of commodities:

4. Commodity i $\qquad i = S, B, A$

If $E_{iY_1} > 0$ i is an ordinary good		If $E_{iY_1} < 0$ i is an inferior good
If $0 < E_{iY_1} < 1$ i is a necessity	If $E_{iY_1} > 1$ i is a luxury	

The same classifications can be applied to T_1.

An unfortunate difficulty remains, as we should like to have Y and T in our

demand functions instead of Y_1 and T_1.[14] This is particularly true for Y since it is usually available in the data and Y_1, though theoretically constructable or obtainable, seems out of reach in most endeavours. We will dispose of T or T_1 in another manner below. If we assume that increases in money income accrue in the same proportion to Y_1 as its initial amount, then results 3 and 4 are unaltered when we replace Y_1 with Y, but results 1 and 2 have no straightforward analogue. A mathematical statement of the assumption for Y is

$$\frac{\Delta Y_1}{\Delta Y} = \frac{Y_1}{Y}$$

$$\frac{\Delta Y_1}{Y_1} = \frac{\Delta Y}{Y}$$

$$\frac{\frac{\Delta Y}{Y}}{\frac{\Delta Y_1}{Y_1}} = 1 \quad : \text{ i.e.,}$$

$$E_{YY_1} = \frac{Y_1}{Y}\frac{\partial Y}{\partial Y_1} = 1$$

Results 3 and 4 hold, since

$$E_{iY_1} = \frac{Y_1}{i}\frac{\partial i}{\partial Y_1} = \frac{Y_1}{i}\frac{\partial i}{\partial Y}\frac{\partial Y}{\partial Y_1} =$$

$$= \frac{Y}{i}\frac{\partial i}{\partial Y}\frac{Y_1}{Y}\frac{\partial Y}{\partial Y_1} = E_{iY}E_{YY_1} = E_{iY}$$

However,

$$\frac{\partial j}{\partial P_i}_{\bar{U}} = \frac{\partial j}{\partial P_i} + i\frac{\partial j}{\partial Y_1}$$

$$= \frac{\partial j}{\partial P_i} + i\frac{\partial j}{\partial Y}\frac{\partial Y}{\partial Y_1}$$

so that, unless $\Delta Y = \Delta Y_1$, the sign and symmetry conditions based on Y_1 do not hold when it is replaced by Y.

This analysis implies that we should employ data on Y_1, expenditure on housing and travel, instead of Y, total income, in the demand function. It might be possible, given some estimates of the signs and magnitudes of $\frac{\partial j}{\partial P_i}$ and $\frac{\partial j}{\partial Y}$, to infer some conditions on $\frac{\partial j}{\partial P_{i\bar{U}}}$, but a correction factor for $\frac{\partial Y}{\partial Y_1}$ should be estimated, too.

A final assumption, which seems necessary to avoid specifying an empirical quantity of leisure time for each consumer, is that all individuals have the same stock of

[14]If we used Y and T as our initial constraints we could obtain demand functions with Y and T as arguments; the difficulty is that it is not obvious that we end up with demand functions which have price arguments from the first commodity group only.

leisure time. This permits us to treat T as part of the constant in a linear statistical model. A case justifying this can be made, casual empiricism notwithstanding. If we define a "normalised" individual as a family consisting of a man, a woman, and some children, we might expect that the two adult members divide their 48 hours in a day so that, after allocation for income-producing work and non-income-producing work (housework, child care, etc.), everyone has about the same residual for meals, sleeping, entertainment and the like. In any case, it seems likely that further research on this point can be postponed.

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