A PROCESS FUNCTION FOR RAIL LINEHAUL OPERATIONS

By Joseph S. DeSalvo*

Like many productive activities, rail freight transportation may be divided into various processes. One division that has been used before is: (1) assembly, (2) linehaul, (3) loading and unloading.[1] This paper is concerned with (2) exclusively. The aim is to develop a production function for the linehaul process (hereafter called a linehaul process function).[2]

Such a process function is interesting in its own right: with it one may investigate various productivity relationships, isoquants, and returns to scale for the linehaul process. The process function may also be used to examine the behaviour of linehaul costs. In addition, it would be possible to determine railway costs if process functions for all three processes were available, or if statistical cost functions were available for those categories of cost for which process functions were not available.[3]

The output of the linehaul process function will be measured in the units *ton-miles per hour*. Ton-miles per hour may be thought of as the product of two terms: (1)

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[1]See, for example, Borts [2].

[2]The term *production function* is usually used to describe a firm's activities. However, much less aggregative productive activities may be described by production functions. Each such activity, or process, will possess its own production function. In order to avoid confusion with the firm "production function," these less aggregative production relations are called *process functions*. This distinction between process and production functions may be found also in Walters [20, pp. 11–14]. An alternative use of the term is found in Manne and Markowitz [16]. To Manne and Markowitz, " 'process analysis' always refers to the construction and use of industry-wide, multi-industry, and economy-wide models which attempt to predict production relationships on the basis of technological structure" [16, p. 4].

[3]An example is provided in DeSalvo and Lave [6].

cargo tonnage and (2) average speed of the train in miles per hour. The analysis involves methods for estimating the two terms to be multiplied.

Cargo tonnage will depend on various characteristics of the vehicle, *e.g.*, size, type, and number of cargo-carrying units. It may also vary with commodity type and with institutional restraints. Average speed between origin and destination, the second term in the process function, is a complicated relationship involving acceleration, deceleration, cruising and delays. This relationship clearly will depend on the motive power of the vehicle, the vehicle drag-force characteristics, the terrain and a variety of delay-causing factors. The linehaul process function, therefore, relates the cargo ton-miles per hour of a vehicle from origin to destination to a host of variables.

The output of the linehaul process function, ton-miles per unit time, reflects the quantity of cargo carried, the distance it is carried, and the speed of the carrying vehicle. It is not an unambiguous measure of output, as is well known, but it has been adopted in many studies. Usually, however, the time unit is the *year* or *quarter* rather than the *hour*. The choice of time unit is dictated by the data and other considerations, as noted in the next paragraph. Since the linehaul process function is developed from engineering data and relationships, the time unit is not restricted by the data. Moreover, many factors affect the speed of the vehicle, and therefore its productivity. A process function that reflects this situation would seem desirable.

It should be remarked, however, that the prediction of the ton-miles generated by a vehicle over a period of time composed of many point-to-point movements should involve consideration of scheduling problems. It would not be correct to assume that each movement was made optimally, since in practice equipment is often not available when and where it should be. If, however, the scheduling conditions were known, the process function could be used to estimate the output for each of the point-to-point movements involved. This might constitute an argument in favour of a longer time unit in the output measure. For example, a statistically estimated linehaul process function whose output was, say, ton-miles per quarter might avoid the problems raised by scheduling considerations.[4]

FORM OF LINEHAUL PROCESS FUNCTION

The process function has the following form:

$$Q = V_e C, \tag{1}$$

in which Q = ton-miles per hour
C = cargo tonnage
V_e = average speed between origin and destination.

If route characteristics are unchanged over the course of the trip, and if all delays as well as acceleration and deceleration are ignored, the process function takes the following form:

$$Q = V_f C, \tag{2}$$

[4] I owe this point to Professor John Kain.

where V_f represents the full speed of the vehicle.[5] This last is the form of the linehaul process function that will be subjected to economic analysis in subsequent sections.

Full-speed operations are typically assumed in railroad engineering textbooks,[6] and discussions with railroad personnel also lead one to believe trains are operated at full speed whenever possible (sometimes even where speed limits would be exceeded). Omission of acceleration and deceleration is probably not very important, because these operations require relatively small amounts of time and cover relatively small proportions of the entire trip distance. In any case, standard formulas exist for predicting acceleration time and distance covered.[7] It would be more difficult, however, to determine the number of stops and starts which would be required to estimate time lost by acceleration and deceleration. Omission of linehaul delays is a much more serious question. Unfortunately, no completely satisfactory delay analysis exists.[8] Were one available, it could easily be integrated into the linehaul process function.[9]

The next section develops the cargo tonnage relationship for a train. Since the total weight of the train affects its speed, a relationship for determining the light (or tare) weight of a train is developed in this section also. In the following section the train full-speed relationship is derived. Subsequent sections present economic analysis of the process function. Since no analyses of acceleration, deceleration, delays, or non-full-speed operations are presented, only the simplest form of the rail freight linehaul process function is used (Equation (2)). Nevertheless, examination of this function should provide insights into rail freight linehaul operations.

CARGO CAPACITY AND LIGHT WEIGHT OF RAIL TRAIN

There are several types of freight cars and many sizes for each type. For each size and type of car, one can determine the cargo capacity in cubic feet, the light weight, and the load limit.[10] Load limits set by the Association of American Railroads (AAR) may not be exceeded and are generally much less than the actual cubic capacity would permit. For example, a 50-foot general box car may have a capacity of 5,000 cubic feet but a load limit of 70 tons. A typical medium density commodity, such as corn, may weigh 45 pounds per cubic foot. This means that if the box car were loaded to capacity the cargo weight would be 112.5 tons, which far exceeds the load

[5]For forms of the process function which allow for route characteristics, acceleration, deceleration and delays, see DeSalvo [5, pp. 4–6].

[6]See the discussion of "plotting a velocity profile" in any transportation engineering textbook, *e.g.*, Hennes and Eske [11, p. 336].

[7]For a discussion of train acceleration and deceleration and some examples of distance travelled and time consumed during these activities, see Hay [9, pp. 134–139; 141–147].

[8]One of the most widely accepted methods for predicting rail linehaul delays is found in Kimball [15]. Also, see [8, pp. 266–271]. Criticisms of the Kimball procedure may be found in Mostafa [17].

[9]See DeSalvo [5].

[10]See Combes [3].

limit. However, a low density commodity like oats, which weighs about 26 pounds per cubic foot, would be limited by space rather than by weight, for the freight car when filled to capacity would weigh 65 tons, less than the load limit of 70 tons.

To put this argument into symbols, the following definitions are needed:

K_{ij} = capacity of ith car for jth commodity, in tons
L_i = load limit of ith freight car, in tons
x_i = cargo volume of ith car, in cubic feet
g_j = weight of jth commodity per unit volume, in tons per cubic foot
k_i = proportion of capacity of ith car utilized, $0 \leqslant k_i \leqslant 1$
c_i = cargo weight of ith car, in tons.

Capacity is defined as follows:

$$K_{ij} = \min (L_i, x_i g_j). \tag{3}$$

Then the cargo weight of the ith car for the jth commodity is

$$c_i = k_i K_{ij}, \; 0 \leqslant k_i \leqslant 1. \tag{4}$$

An example will illustrate the above relationships. Suppose, for a given freight car and a given commodity,

L_i = 70 tons
g_j = .01 tons/cu. ft.
x_i = 5,000 cu. ft.

Then K_{ij} = min (70 tons, 50 tons) = 50 tons; *i.e.*, the car is space limited for the assumed commodity, and the cargo weight of the car is given by Equation (4), where k_i refers to the fraction of the ith car's capacity utilized. If another, denser, commodity is considered, say the rth commodity, where g_r = .025, then K_{ir} = min (70, 125) = 70, and the car is weight limited. The cargo weight is again given by Equation (4).

The cargo weight of a train of freight cars will be the sum of the cargo weights of the individual freight cars comprising the train, or, in symbols,

$$C = \sum_{i=1}^{N_c} c_i = \sum_{i=1}^{N_c} k_i K_{ij}, \tag{5}$$

where N_c is the number of cars in the train and j represents the commodity with which the car is loaded. This formulation allows for N_c different cars, N_c different commodities (but only one to a car), and N_c different utilization rates. If all cars were identical and hauled the same amount of the same commodity, and if the cars were weight-limited for that commodity, the cargo weight of a train would be given by

$$C = N_c k L. \tag{6}$$

In addition to the cargo weight of a train, it will be necessary in the subsequent analysis to know the tare or light weight of the cars and the gross weight of the locomotive. Again, for any given freight car, one can easily determine the tare weight of the car, w_{t_i}. Therefore, for any given train of cars, one can determine the light weight of the train of cars,

$$W_t = \sum_{i=1}^{N_c} w_{t_i}, \tag{7}$$

where N_c is the number of cars in the train. The gross weight of one freight car is

$$w_{c_i} = c_i + w_{t_i}\,, \tag{8}$$

and the gross weight of the train of cars is

$$W_c = C + W_t\,. \tag{9}$$

The gross weight of a locomotive may be similarly obtained.[11] But the gross weight of a locomotive seems to be related to its horsepower and number of axles, *i.e.*, for a given number of axles (4, 6 or 8) the weight of the locomotive is an increasing function of horsepower. And, for any given horsepower, weight is an increasing function of number of axles. In the subsequent analysis it will prove convenient to have a functional relationship between horsepower, weight and axles. The following relationship will be used:

$$w_L = -3.62 + \underset{(.00194)}{.00834HP_L} + \underset{(1.72)}{27.14a_L} \qquad R^2 = 0.98 \tag{10}$$

in which w_L = weight of locomotive unit in tons
HP_L = horsepower of locomotive unit
a_L = number of axles of locomotive unit.

Both coefficients are significantly different from zero. The t-values for horsepower and axles are, respectively, 4.3 and 15.7. Standard errors are given in parentheses below the estimates to which they relate. The coefficient of determination is 0.98. This relationship was obtained by regression analysis applied to twenty-four sample observations of diesel-electric locomotives contained in *Car and Locomotive Cyclopedia*. It should prove applicable to present-day United States and Canadian diesel-electric locomotives.

Horsepower may be added to a train in one of two ways: (1) by using a larger horsepower locomotive or (2) by using additional locomotives. For example, a motive force of 6,000 horsepower may be obtained from one 6,000-horsepower locomotive or from four 1,500-horsepower locomotives. It would be interesting to be able to investigate these alternative methods of obtaining horsepower. This can be allowed for by defining locomotive gross weight, W_L, as

$$W_L = \sum_{i=1}^{N_L} w_{L_i}\,, \tag{11}$$

where N_L is the number of locomotive units and w_{L_i} is the weight of the ith locomotive unit. If $N_L = 1$, then $W_L = w_L$. This formulation will permit investigation of the effect of changing horsepower on the ton-miles per hour generated by the train, where horsepower can be changed either by adding locomotives or by using a locomotive of greater horsepower.

The gross weight of the train is expressed as

$$W = W_L + W_c = \sum_{i=1}^{N_L} w_{L_i} + \sum_{i=1}^{N_c} c_i + \sum_{i=1}^{N_c} w_{t_i}\,. \tag{12}$$

The preceding discussion has developed a formulation of the cargo and gross

[11] *Ibid.*

weight of a train. Both of these are components of the train process function, for the cargo tonnage transported by the train certainly affects the ton-miles per hour produced by the vehicle, and the gross weight of the train affects the speed at which it may travel. The next section begins the analysis of a train in motion.

ANALYSIS OF TRAIN IN MOTION

The cruising speed of a vehicle depends on the resistance characteristics or drag force acting on the vehicle and the tractive-effort or motive properties of the propulsion system. These forces are both functions of velocity as well as vehicle characteristics and roadway characteristics. The equation of motion for the vehicle is

$$F = F_t(v) - F_d(v) = m\,(dv/dt)\,, \tag{13}$$

in which

F = resultant force
$F_t(v)$ = tractive force
$F_d(v)$ = drag force
dv/dt = acceleration
m = mass[12].

Equation (13) is a form of Newton's second law of motion: force is the product of mass and acceleration. An alternative statement is

$$dv/dt = [F_t(v) - F_d(v)]/m. \tag{14}$$

If the vehicle is operating at full velocity, V_f, *i.e.*, the maximum attainable velocity for a given power input, then $dv/dt = 0$, and it may be possible to solve Equation (13) for $v = V_f$.

In the case of rail trains, as will be developed more fully below, the drag-force function is quadratic in velocity, but the tractive-effort function is a rectangular hyperbola, *i.e.*,

$$F_d(v) = A_0 + A_1 v + A_2 v^2 \tag{15}$$

and

$$F_t(v) = B/v. \tag{16}$$

Therefore, the equation of motion for the system is

$$dv/dt = [(B/v) - (A_0 + A_1 v + A_2 v^2)]/m. \tag{17}$$

When $dv/dt = 0$, Equation (17) reduces to the following cubic equation:

$$v^3 + k_2 v^2 + k_1 v + k_0 = 0\,, \tag{18}$$

where $k_0 = -B/A_2$, $k_1 = A_0/A_2$, and $k_2 = A_1/A_2$ (note that m has dropped out of the equation). This cubic equation may be solved for velocity. As will be shown below, one of the three resulting roots will be real and positive. The real, positive root is identified with the full speed of the train, V_f, and the other roots have no physical significance.

The following subsections present detailed discussions of the train tractive-effort and drag-force functions.

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[12]This notation is from Haase and Holden [7].

Tractive-Effort Function

A general tractive-effort function for diesel-electric locomotives is derivable from the definition of horsepower. Moreover, it takes account of mechanical and electrical losses from auxiliary units of the locomotive. This function is [13]

$$F_t(v) = 375\,(HP - HP_a)\,(e/v), \tag{19}$$

in which

$F_t(v)$ = tractive effort, in pounds
HP = total rated horsepower of locomotive units
HP_a = horsepower used by auxiliaries
e = an efficiency factor, mechanical-electrical, taken as 82.2 per cent
v = velocity, in miles per hour.

When these factors are given numerical values from conventional design and operation, the tractive-effort equation reduces to

$$F_t(v) = (308HP)/v. \tag{20}$$

Drag-Force Function

The most commonly used drag-force function in railroad engineering is the Davis formulation [4].[14] The relationship is usually presented as follows for both rail cars and locomotives:

$$F_d(v) = (1.3 + 29/w + bv + SXv^2/wa)\,wa, \tag{21}$$

in which

$F_d(v)$ = drag-force, or resistance of car or locomotive, in pounds
w = weight per axle of car or locomotive, in tons
a = number of axles
b = coefficient of flange friction = $\begin{cases} 0.045 \text{ for freight cars} \\ 0.03 \text{ for locomotives} \end{cases}$
S = drag coefficient of air = $\begin{cases} 0.0025 \text{ for locomotives} \\ 0.0005 \text{ for freight cars} \end{cases}$
X = cross-sectional area = $\begin{cases} 105\text{–}120 \text{ sq. ft. for locomotives} \\ 85\text{–}\ 90 \text{ sq. ft. for freight cars.} \end{cases}$

Frontal area could be determined for each locomotive and car; but, since the subsequent analysis is to be applied to a large variety of cars and locomotives and since the range of variation of X is small, it will be assumed to be equal to 112.5 for locomotives and 87.5 for freight cars (the arithmetic averages of the end points of the ranges of variation).

Equation (21) may be put into a more useful form if the implied multiplication of the bracketed expression by wa is carried out. Note that wa is the gross weight of the vehicle. Substituting the values of b, S, and X, two drag-force functions are obtained, one for a locomotive and one for a freight car, *viz.*,

$$F_d^L(v) = 1.3w_L + 29a_L + .03w_Lv + .28125v^2 \tag{22}$$

[13]Hay [8, pp. 200–201].
[14]See Hay [8, pp. 174–175].

and

$$F_d^c(v) = 1.3w_c + 29a_c + .045w_cv + .04375v^2, \quad (23)$$

in which

$F_d^L(v)$ = drag-force of locomotive, in pounds
$F_d^c(v)$ = drag-force of freight car, in pounds
w_L = gross weight of locomotive, in tons
w_c = $c + w_t$ = gross weight of freight car, in tons
a_L = number of axles of locomotive
a_c = number of axles of freight car
v = velocity of locomotive or car, in miles per hour.

The above formulas give resistance of only one locomotive and one freight car. Total train resistance is the sum of the locomotive resistance and the resistances of all the freight cars or

$$F_d(v) = \sum_{i=1}^{\mathcal{N}_L} F_{d_i}^L(v) + \sum_{i=1}^{\mathcal{N}_c} F_{d_i}^c(v), \quad (24)$$

where

$F_d(v)$ = total train resistance
$F_{di}^L(v)$ = locomotive resistance of ith unit
$\mathcal{N}_L$ = number of locomotive units in train
$F_{di}^c(v)$ = freight car resistance of ith car
$\mathcal{N}_c$ = number of freight cars in train.

Upon substituting Equations (22) and (23) into Equation (24), one obtains the drag-force function for a train as follows:

$$F_d(v) = 1.3W + 29A + (.03W_L + .045W_c)v + (.28125\mathcal{N}_L + .04375\mathcal{N}_c)v^2, \quad (25)$$

in which

$W = W_L + W_c$

$W_L = \sum_{i=1}^{\mathcal{N}_L} W_{L_i}$ = gross weight of locomotive units

$W_c = \sum_{i=1}^{\mathcal{N}_c} w_{c_i}$ = gross weight of freight cars

$A = A_L + A_c$

$A_L = \sum_{i=1}^{\mathcal{N}_L} a_{L_i}$ = total number of axles of locomotive units

$A_c = \sum_{i=1}^{\mathcal{N}_c} a_{c_i}$ = total number of axles of freight cars.

If there is only one locomotive unit and all freight cars are identical, then the drag-force function reduces to

$$F_d(v) = 1.3(w_L + \mathcal{N}_c w_c) + 29(a_L + \mathcal{N}_c a_c) + (.03w_L + .045\mathcal{N}_c w_L)v + (.28125 + .04375\mathcal{N}_c)v^2. \quad (26)$$

Adjustment for Grade and Curvature

It is not difficult to adjust the preceding drag-force function to account for the effect of grade. Let the extra resistance on the train due to grade be denoted by F_s, which may be negative or positive according to whether the train is going downhill or uphill. F_s is equal to 20 pounds per ton per percentage of grade.[15] For example, a train climbing a 2 per cent grade encounters additional resistance equal to 40 pounds per ton of train weight. The relationship between grade resistance, train weight, and percentage grade may be expressed as follows:

$$F_s = 20Ws, \tag{27}$$

in which

F_s = grade resistance, in pounds
W = total weight of train, engine and cars, in tons
s = percentage gradient of terrain.

In order to obtain total train drag-force including grade resistance, one need merely add the grade resistance to the train resistance, *i.e.*,

$$F_{ds}(v) = F_d(v) + F_s, \tag{28}$$

or

$$F_{ds}(v) = (1.3 + 20s)W + 29A + (.03W_L + .045W_c)v + (.28125N_L + .04357N_c)v^2 \tag{29}$$

where $F_{ds}(v)$ means drag force adjusted for gradient of terrain, and the other variables have the same meanings as before. Of course, if the train is operating on level ground $s = 0$, and Equation (29) reduces to Equation (25).

Track curvature is another important factor affecting the performance of freight trains. When a train rounds a curve, resistance is encountered in addition to train resistance and grade resistance. This additional resistance due to track curvature is known as curve resistance. There are several theories to explain why curve resistance occurs; it is unnecessary to discuss this here.[16] In any case, it is easy to adjust the resistance relationship for the additional resistance of curvature.

On the basis of railroad tests, the American Railway Engineering Association (AREA) [1] has adopted a recommended value for curve resistance of 0.8 pounds per ton per degree of curvature. The degree of curvature is the amount of central angle subtended by a chord of 100 feet. Total curve resistance, F_c, for a train which has a gross weight of W tons rounding a curve of c degrees is given by

$$F_c = 0.8Wc. \tag{30}$$

Curve resistance, like grade resistance, is an addition to the resistance function. The resistance function adjusted for curve and grade is, therefore,

$$F_{dsc}(v) = F_d(v) + F_s + F_c = (1.3 + 20s + .8c)W + 29A + (.03W_L + .045W_c)v + (.28125N_L + .04375N_c)v^2. \tag{31}$$

[15]For a derivation of the grade-adjustment factor, see Hay [9, pp. 82–83].
[16]But see Hay [9, pp. 52–55].

Determination of Train Cruising Speed

The equation of motion for a rail train operating at cruising speed is

$$F_t(v) - F_{dsc}(v) = 0. \tag{32}$$

Substituting the functional forms for F_t and F_{dsc}, one obtains the following cubic equation:

$$308HP - [(1.3 + 20s + .8c)\ W + 29A]v - (.03W_L + .045W_c)v^2 - (.28125\mathcal{N}_L + .04375\mathcal{N}_c)\ v^3 = 0. \tag{33}$$

Solving this cubic equation for velocity will yield the cruising speed, V_f, of a rail train as a function of HP, s, c, W, A, W_L, $\mathcal{N}_L$ and $\mathcal{N}_c$. Fortunately, a solution to a cubic equation of this form exists, and a standard computer program is available for obtaining it. See Appendix.

RAIL PRODUCTIVITY RELATIONSHIPS

Given the relationships developed in the last two sections, it is now possible to investigate the rail linehaul process function, ignoring route segmentation, acceleration, deceleration, and linehaul delays. Route segmentation, by its very nature, is specific to a given trip. Ignoring it in the following analysis does not preclude examination of the effects of route characteristics on the output of a train. Acceleration and deceleration may usually be safely ignored. Delays en route are important but will be omitted nevertheless. The form of the linehaul process function which omits acceleration, deceleration, route segmentation and delays is Equation (2), repeated here:

$$Q = V_f C, \tag{34}$$

where

Q = output of the train, in ton-miles per hour
V_f = cruising speed of the train, in miles per hour
C = cargo weight of the train, in tons.

Each of the terms V_f and C represents a function. In general, these functions may be specified as follows:

$$V_f = f(HP_{L_i}, \mathcal{N}_L, a_{L_i}, s, c, \mathcal{N}_c, a_{c_j}, k_j, x_j, g_j) \tag{35}$$

$$C = g\ (\mathcal{N}_c, k_j, L_j, x_j, g_j) \tag{36}$$
$$i = 1, \ldots, \mathcal{N}_L$$
$$j = 1, \ldots, \mathcal{N}_c$$

in which

HP_{L_i} = horsepower of ith locomotive unit
$\mathcal{N}_L$ = number of locomotive units
a_{L_i} = number of axles per ith locomotive unit
s = gradient of terrain, as percentage
c = degree of curvature, in degrees
$\mathcal{N}_c$ = number of cars

$a_{c\,j}$ = number of axles per jth car
k_j = proportion of the jth car loaded, $0 \leqslant k_j \leqslant 1$
L_j = load limit of jth car
x_j = cargo volume of jth car, in cubic feet
g_j = weight of jth commodity per unit volume, in tons per cubic foot.

Note that this formulation is quite general, allowing for almost any conceivable combination of the inputs. To simplify matters somewhat, the following assumptions are made. (1) All freight cars in the train are identical. (2) All locomotive units on a given train are identical. (3) Every freight car in a given train contains the same amount of cargo weight. (4) Freight cars are weight-limited rather than space-limited; another way of stating this is that the commodities being transported are so dense that the load limit of a freight car is reached before its cubic capacity is exhausted.[17] These assumptions permit the following simplification of the speed and cargo functions:

$$V_f = h\ (HP_L, N_L, a_L, s, c, N_c, a_c, k, L) \tag{37}$$

$$C = N_c k L. \tag{38}$$

With the aid of a production function, one may examine the effect of each individual input on the output of the productive unit, given constant levels of the other inputs. One is concerned with the total, average, or marginal productivity of the productive unit with respect to a particular input. In addition, the possibilities of substitution between inputs may be investigated through the production function; that is, isoquants may be determined. From these and the related input costs, expansion paths and cost functions may be obtained.

These properties of a production function may be determined analytically if the production function relation is simple enough. That is, functional forms may be obtained for such things as productivity curves, isoquants, and expansion paths. In the case of the rail linehaul process function developed above, this analytical treatment is precluded because the relationship is too complex. (Recall that the function V_f results from the solution of a cubic equation.) The alternative method of obtaining the desired results is to characterize the function numerically, using tables and graphs. This is the procedure followed below.

The most important effects to be examined seem to be the effect of the locomotive input and the effect of the car input on train performance. The most important input substitution to consider is that between locomotive power and cars. In addition to these relationships it is also of interest to investigate the effect on train performance of terrain, as represented by gradient and curvature. Finally, one might like to know the effect of less-than-carload shipments. All these are discussed below.

First, the relationship between the locomotive input and the train output is

[17] An additional assumption, of less importance, is that the number of axles on each locomotive is six. As was shown in Equation (10), the number of axles is related to the weight of the locomotive. The above assumption tends to overestimate the weight of locomotives with horsepower ranging from 500 to 2,000, since these usually have four axles, and to underestimate the weight of locomotives with horsepower ranging from 4,500 to 6,000, since these usually have eight axles.

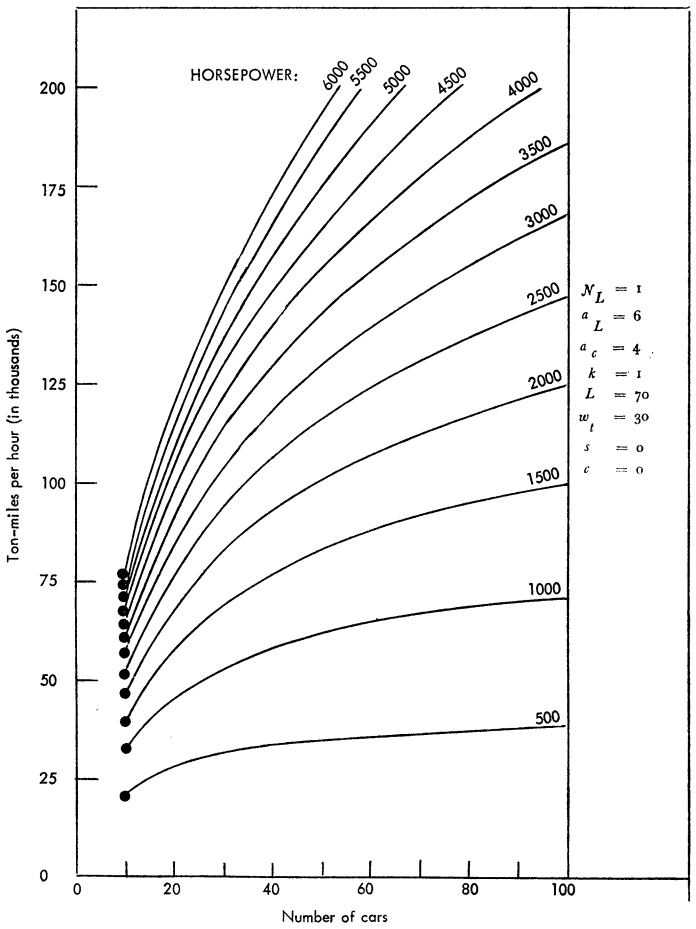

Figure 1. *Productivity Curves for Rail Linehaul Process Function*

investigated. The locomotive input in the process function is represented by three variables: (1) horsepower of a locomotive unit, (2) number of axles of a locomotive unit, and (3) number of locomotive units. It is total horsepower of the motive unit(s) that determines the tractive effort of the vehicle. (See Equation (20).) Under the assumptions stated above, horsepower may be obtained either from a single locomotive unit or from several locomotive units of the same horsepower. (Unequally powered locomotive units for a given train are excluded by assumption.)

Figure 1 displays total productivity of cars for various levels of horsepower. The values of the variables entering the process function are given in the box accompanying the figure. They represent one locomotive pulling standard, fully loaded 70-ton box cars over straight level track. These cars have a light weight of 30 tons each. The following observations are made: (1) the total productivity curves for fixed horsepower are strictly concave, *i.e.*, the marginal productivity of cars is monotone

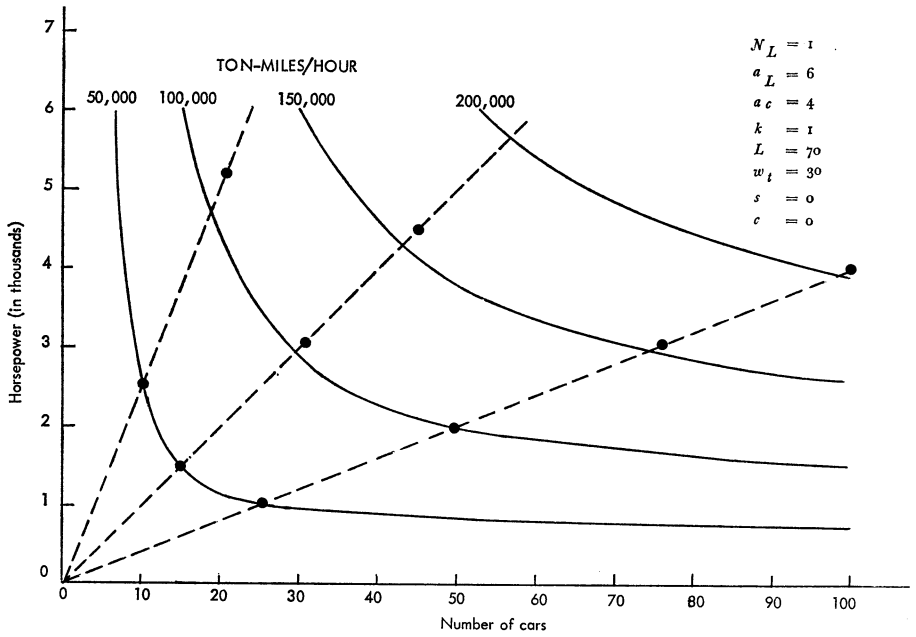

Figure 2. *Isoquants of Rail Linehaul Process Function*

decreasing, or $(\partial^2 Q/\partial N_c^2) < 0$; (2) the marginal productivity of horsepower diminishes. This is shown in Figure 1 by the curves getting closer together as one moves upward from a given number of cars, *i.e.*, $(\partial^2 Q/\partial HP^2) < 0$; (3) the marginal products of both inputs remain positive within the ranges considered, *i.e.*, $(\partial Q/\partial HP) > 0$ and $(\partial Q/\partial N_c) > 0$ for $500 \leqslant HP \leqslant 6{,}000$ and $10 \leqslant N_c \leqslant 100$. It is likely that the marginal product of cars would become negative at some point beyond a train length of 100 cars, at least for the low-horsepower trains. The positivity of the marginal products means that as the train gets larger and larger the resulting decrease in speed is more than offset by the resulting increase in additional carrying capacity.

The productivity curves of Figure 1 also permit the examination of substitution possibilities between the car and locomotive inputs. By selecting a particular rate of output and drawing at that level of output a horizontal line across the family of total product curves, one obtains the combinations of the two inputs yielding the given level of output. The curve drawn through such a collection of points is called an isoproduct curve or isoquant.

By examining the isoquants shown in Figure 2, a couple of interesting properties of rail transportation may be determined. The isoquants are negatively sloped throughout, indicating that the two inputs are technical substitutes for one another, *i.e.*, if fewer cars are used, more horsepower must be used in order that total output may remain unchanged. Moreover. the isoquants are convex to the origin; this means that, while the two inputs are technical substitutes, they are not perfect substitutes. More cars will just compensate for smaller and smaller amounts of horsepower, and vice versa.

Finally, isoquants can be used to investigate the returns to scale exhibited by the productive process. *Returns to scale* describes the output response to a proportionate increase in all inputs. If output increases in the same proportion, returns to scale are constant. They are increasing if output increases in greater proportion, and decreasing if it increases in smaller proportion, than the increase in inputs.[18] There are two equivalent ways of determining which of these three categories prevails when an isoquant graph is available. In both methods straight lines must be drawn from the origin of the horsepower-car axes into the plane. Three of these are shown in Figure 2. Upon measuring the distance along a line from the origin between points on successive isoquants which increase by a constant amount, one will find that these distances become smaller. This shows that a given increment in output may be obtained with a proportionately smaller increase in both inputs, a condition of increasing returns to scale. This condition prevails along all the lines drawn in Figure 2. Alternatively, one could note the points along a line from the origin which represent given proportionate increases in inputs. These are shown in Figure 2 by dots along the three straight lines. The first set of dots were selected to lie on the isoquant representing 50,000 ton-miles per hour. Successive sets of dots represent, respectively, increases of 100 per cent, 50 per cent, and $33\frac{1}{3}$ per cent. It can be observed that the isoquants representing increases in output of 100, 50, and $33\frac{1}{3}$ per cent lie below the corresponding points on the line from the origin. Moreover, the isoquants lie increasing distances below these points as the rate of output increases. Again, this indicates increasing returns to scale, because proportionate increases in all inputs are accompanied by greater than proportionate increases in output.

A qualification of these results should be voiced at this point. It was assumed in the above calculations that only one locomotive unit was employed. Motive power was added to a train by using a more powerful locomotive. The rail process function allows for the possibility of multiple locomotive units, and the assumption of a single unit was made only for simplicity. Multiple units would typically be used when total motive power exceeded 6,000 horsepower, for at present the largest locomotive units are of that size; but it is conceivable that multiple locomotive units would be employed even where the total horsepower could be obtained from one unit. This could occur for two reasons. First, scheduling problems might dictate the use of multiple units where a single unit would be more efficient. Second, in certain circumstances multiple units might be more efficient than a comparable single unit. This applies when the total weight of the multiple units is less than that of a comparable single unit, and when the total number of axles of the multiple unit is less than or equal to that of a single unit. Both weight and number of axles increase train resistance (see Equation (31)); thus increases in these factors, *ceteris paribus*, decrease output. Normally, however, for total horsepower less than 6,000 single locomotive units will be more efficient than multiple units. In any case, the process function may be used to rule out inefficient alternatives. Table 1 illustrates how certain multiple units compare with their corresponding single unit locomotives in pulling a given train.

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[18]See Henderson and Quandt [10, p. 62].

TABLE 1

Effect of horsepower on train performance for single and multiple locomotive units for 50-car train

Total Horsepower	*Ton-miles per Hour*	
	Multiples of 2,000-HP (4-axle locomotives)	*Single Unit*
2,000	101,374	101,374 (4 axles)
4,000	152,765	155,108 (6 axles)
6,000	187,600	193,909 (8 axles)

The effect of gradient on train performance is illustrated by Table 2. The train whose output is examined in that table consists of 50 fully loaded, 70-ton freight cars pulled by one 2,000-horsepower locomotive. The table ends with a gradient of 3 per cent because few gradients are steeper than that.[19]

TABLE 2

Effect of gradient on train performance for fifty 70-ton box cars pulled by 2,000-horsepower locomotive

Gradient (per cent)	*Speed (miles/hour)*	*Output (ton-miles/hour)*
0	28.96	101,374
.5	9.31	32,600
1.0	5.30	18,540
1.5	3.69	12,903
2.0	2.82	9,885
2.5	2.29	8,009
3.0	1.92	6,731

Curvature, the other terrain factor to be considered, also impedes train performance, although less dramatically. Table 3 illustrates the effect of curvature on performance of the same train. Curvature may go as high as 30 to 40 degrees, but curvature above 20 degrees is not recommended.[20] Curvatures of more than 5 degrees are not considered in the table.

The final factor to be investigated is the effect of loading on train performance. It is common knowledge that less-than-full car loadings are discriminated against in rail transport pricing. While this is surely a question of relative costs, it is of interest to look at this question from the production side in order possibly to obtain some evidence on it. The process function permits investigation of this point, for it has built into it a variable representing the loading characteristics of a train. Although the simplifying assumptions given earlier require that all cars be loaded to the same

[19]Hay [8, p. 221].
[20]*Ibid.*, p. 445.

TABLE 3

Effect of curvature on train performance for fifty 70-ton box cars pulled by 2,000-horsepower locomotive

Curvature (degrees)	*Speed (miles/hour)*	*Output (ton-miles/hour)*
0	28.96	101,374
.5	27.18	95,137
1.0	25.54	89,384
1.5	24.03	84,092
2.0	22.64	79,235
2.5	21.37	74,784
3.0	20.20	70,706
3.5	19.13	66,970
4.0	18.16	63,546
4.5	17.26	60,404
5.0	16.43	57,517

extent, the process function permits investigation of any conceivable loading condition, if desired.

Table 4 shows, for the same train considered above, what happens to speed and output as loading varies from empty to full.

TABLE 4

Effect of carloading on train performance for fifty 70-ton box cars pulled by 2,000-horsepower locomotive

Proportion Loaded (percent)	*Speed (miles/hour)*	*Output (ton-miles/hour)*
0	40.90	0
10	39.25	13,738
20	37.74	26,416
30	36.34	38,156
40	35.05	49,063
50	33.85	59,229
60	32.73	68,734
70	31.67	77,644
80	30.72	86,021
90	29.81	93,916
100	28.96	101,374

Although output increases rapidly as the cars become fuller, there is diminishing marginal productivity associated with the factor "proportion loaded." If hourly operating costs of the train were constant no matter what the load, then both unit cost (*i.e.*, cost per ton-mile) and marginal cost would be decreasing throughout. This would constitute an argument in favour of full carloading.

RAIL COSTS

In the preceding sections of this paper a rail linehaul process function has been developed, some properties of which were investigated in the last section. The process function can help to determine the behaviour of rail linehaul costs. The costs of the linehaul process are investigated by deriving a family of short-run average cost curves (analogous to the plant unit cost curves of a firm). Here, however, the "plant" is a railroad freight train, and plant "scale" varies as the size of the locomotive changes. Economies of scale are indicated by the envelope to the short-run unit cost curves.

Costs of Linehaul Process

Aside from depreciation and interest, the primary linehaul costs are crew wages and fuel expenses. Others, such as lubricants and water, are so small that they can be safely ignored.[21] In order to determine the train's cost curves, methods of estimating the individual cost components must be developed.

A typical freight train crew consists of an engineer, a conductor, and two brakemen.[22] Their wages are based on the distance travelled. Time is converted into distance on the assumption of an average freight-train speed of 12.5 miles per hour. An eight-hour period is therefore equivalent to a 100-mile trip. In order to determine the crew cost of a trip, the first 100 miles is charged at one rate and the remaining miles at a lower rate. These mileage rates are effectively independent of the actual trip time involved. If, however, the train actually attains an average speed for the trip of less than 12.5 miles per hour, "overtime" is paid on the basis of the miles that could have been travelled at a speed of 12.5 miles per hour. Thus, if a 125-mile trip which "should" have taken 10 hours actually takes 12 hours, it is equivalent to a 150-mile trip for crew-wage purposes, the additional 25 miles being charged at the higher rate. These conditions may be stated algebraically as follows:

$$C_W = \begin{cases} r_1 D, & \text{if } D \leqslant 100 \text{ and } T \leqslant D/12.5 \\ 100 r_1 + r_2 (D - 100), & \text{if } D > 100 \text{ and } T \leqslant D/12.5 \\ 12.5 r_1 T, & \text{if } D \leqslant 100 \text{ and } T > D/12.5 \\ 100 r_1 + r_2 (12.5 T - 100), & \text{if } D > 100 \text{ and } T > D/12.5 \end{cases} \tag{39}$$

where

C_W = crew member's cost to the trip,
r_1 = crew member's wage rate per mile for first 100 miles,
r_2 = crew member's wage rate per mile after first 100 miles,
D = actual trip distance in miles,
T = actual trip time in hours.

All crew members are paid on the formula set out above, though there are differences

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[21] Of five linehaul expense categories – enginemen, trainmen, fuel, water and lubricants – water and lubricants constituted 1.7 per cent of the total. See Interstate Commerce Commission [14, Part 1, p. 93].

[22] The description of train crew make-up, method of payment and wage rates contained in this paper was obtained from a Midwestern railroad.

in their rates. In addition, the rates vary with other factors. An engineer receives a higher wage rate for operating two engine units rather than one, and a higher rate still if there are three engine units on the train. The wage rates of the conductor and brakemen rise with the number of cars in the train. The relationships are summarized in the following table: the figures are those used by a Midwestern railroad in July 1967.

TABLE 5
Freight train crew wage rates, 1967

Crew Member	*Rates/Mile for First 100 Miles*	*Rates/Mile after First 100 Miles*
Engineer		
1 engine unit	$ .2564	$ .2389
2 engine units	.2657	.2482
3 engine units	.2775	.2575
4 engine units	.2840	.2665
Conductor		
1–81 cars	.2216	.2041
82–105 cars	.2251	.2076
106–125 cars	.2291	.2116
126–145 cars	.2316	.2141
146–165 cars	.2326	.2151
over 165 cars	*a*	*a*
Brakeman		
1–81 cars	.2104	.1860
82–105 cars	.2139	.1895
106–125 cars	.2179	.1935
126–145 cars	.2204	.1960
146–165 cars	.2214	.1970
over 165 cars	*a*	*a*

a $.20 for each additional block of 20 cars or portion thereof.
SOURCE: Midwestern railroad which asked not to be identified.

The contribution of one crew member to the trip cost is given in general by Equation (39). If the trip is 100 miles in length and the crew consists of an engineer, a conductor and two brakemen, the total crew costs for the trip are defined as follows:

1. If train speed is less than 12.5 miles per hour, crew cost is given by
   (a) 1–81 cars: $11.235T
   (b) 82–105 cars: 11.366T
   (c) 106–125 cars: 11.516T
   (d) 126–145 cars: 11.610T
   (e) 146–165 cars: 11.648T
   (f) 166–185 cars: 11.723T
   (g) 186–205 cars: 11.798T
2. If train speed is greater than or equal to 12.5 miles per hour, crew cost is given by:

(a) 1–81 cars: $89.88
(b) 82–105 cars: 90.93
(c) 106–125 cars: 92.13
(d) 126–145 cars: 92.88
(e) 146–165 cars: 93.18
(f) 166–185 cars: 93.78
(g) 186–205 cars: 94.38

Fuel cost is the second major component of train expenses. The amount of fuel consumed on a trip is primarily a function of horsepower, thermal efficiency of the engine units, and energy content of the fuel. A formula is developed for predicting train fuel consumption per hour. If this and the price per gallon of fuel are known, the fuel cost per hour may be calculated.

The AREA *Manual* [1, p. 16–3–35] states that the thermal efficiency, defined as the ratio of work performed to energy consumed, of rail diesel engines is 25–30 per cent. This may be stated in symbols as $W/E = 0.25$, where W is work performed, E is energy consumed and the low end of the range is used. In order that W and E may be in the same units, work, defined as the product of horsepower and time, is converted to Btu. by the following relation: 2,545 Btu. = 1 hp-hr. Since there are approximately 139,000 Btu. in a gallon of diesel fuel,[23] the energy content of N gallons is $E = 139{,}000N$. Thus, the number of gallons consumed per hour, n, by a diesel engine of HP horsepower is obtained from the following expression:

$$.25 = \frac{2545HP}{(139{,}000)\,n,} \tag{40}$$

which when solved for n yields

$$n = 0.073HP. \tag{41}$$

Equation (41) gives the number of gallons per hour that will be consumed by engine units generating HP horsepower. Given the price p per gallon of fuel oil, fuel cost per hour (C_f) is obtained as

$$C_f = p \cdot n = 0.073\,pHP. \tag{42}$$

Fuel cost per trip may be found by multiplying trip time in hours by the fuel cost per hour. If it is assumed that $p = \$0.10$, *i.e.*, that diesel fuel is $0.10 per gallon, then fuel cost per trip is given by

$$F = \$0.0073\text{HP}\cdot\text{T}. \tag{43}$$

This expression is true regardless of the rated horsepower of the engine units; but, since the horsepower actually generated by the locomotive units over the entire trip cannot be known, it will be assumed to be equal to the rated horsepower of the engine units. This means that the train is assumed to operate at full power during those periods when it is moving at all. Since trains utilize full throttle whenever possible, this assumption should not do too much violence to the facts.[24]

[23]The Btu. content of a gallon of diesel fuel was obtained from the marketing department of the American Oil Company's Chicago office. The range of variation was given as 135,680 to 141,700 Btu per gallon.

[24]However, since fuel cost per trip increases as trip time decreases, full-throttle speed will not result in minimum fuel cost per trip.

The next cost to be associated with linehaul operations is depreciation of equipment. Cars and locomotives depreciate partly through use and partly through the passage of time, but no distinction is made between these two causes. Moreover, for convenience, straight-line depreciation is assumed. While this assumption is justified only under certain circumstances,[25] it is quite common accounting practice.

Table 6 shows freight car prices by car types. These figures are averages of actual purchase prices reported to the ICC by railroads. Prices will vary for cars of a given type as size and other factors vary, but these figures should serve where more exact ones are not available.

The following equation represents diesel locomotive prices as a function of horsepower. The equation is a least-squares regression based on twenty-two sample observations from ICC data [13]. Locomotive cost (C_L) is in thousands of dollars.

$$C_L = -4.313 + \underset{(0.007)}{0.094HP_L}, \; R^2 = .90 . \qquad (44)$$

Again, better information could be obtained for specific locomotive types, but estimates based on Equation (44) should be fairly accurate.

TABLE 6
Freight Car Purchase Prices by Types, 1964

Type of Car	*Cost*
Box (general service)	$ 13,083
Box (special service)	17,534
Flat	15,462
Stock	11,546
Gondola	12,504
Hopper (open)	10,380
Hopper (covered)	14,073
Rack	12,000
Refrigerator	21,914
Tank	19,339
Caboose	17,759

SOURCE: Interstate Commerce Commission, *Seventy-Eighth Annual Report on Transport Statistics in the United States for the Year Ended 31 December 1965* (Washington, D.C.: U.S. Government Printing Office, 1965).

Equipment life of 15 years and salvage value of 10 per cent of purchase price are assumed. Straight-line depreciation based on the purchase price less salvage value is used as the estimate of current depreciation expense per annum. This figure is converted to a per-hour basis by dividing the annual figure by the number of hours in a year (8,760).

Depreciation expense per trip for a given train would be calculated as follows: (1) determine number and horsepower of engine units, (2) determine number and type

[25] See Smith [19, p. 109].

of cars, (3) estimate purchase price for each locomotive unit from Equation (44) and for each car from Table 6, (4) subtract 10 per cent of this amount to obtain depreciable portion of the investment, (5) divide the resulting figure by 15 × 8,760 to obtain the depreciation charge per hour, (6) determine time required for the trip from the analysis of train operations presented above, and (7) multiply this figure by the per-hour depreciation charge to get total depreciation charge allocable to the trip.[26]

In particular, if it is assumed that the cars comprising the train are each identical fully loaded general service box cars, the car depreciation charge allocable to the trip is given by

$$D_c = \frac{[\$13{,}083 - .10(13{,}083)]T \cdot N_c}{(15)\ (8760)} = \$.0896 T \cdot N_c \qquad (45)$$

where D_c = total car depreciation per trip
T = actual trip time, in hours
N_c = number of cars.

A similar calculation produces the locomotive depreciation allocable to the trip. The purchase price of a locomotive is estimated via Equation (44). Given this quantity, one may determine the depreciation charges per locomotive per trip, D_L, as

$$D_L = \frac{[(\$94HP - \$4313) - .10(\$94HP - 4313)]T}{(15)\ (8760)} = (\$.0007HP - \$.0295)T. \qquad (46)$$

The final cost to be allocated to the linehaul process is interest on investment in road equipment. Here, as with depreciation, the total interest charge over the life of the equipment will be reduced to a per-hour amount and allocated to the linehaul process on the basis of the time required to complete the trip.

The total interest charge incurred over the life of the equipment will be estimated by the following relation:

$$I = \tfrac{1}{2}WLr\,, \qquad (47)$$

where I = total interest charge
W = purchase price of unit of capital equipment
L = length of life of unit of capital equipment
r = annual interest rate.

Equation (47) gives the total interest paid (or foregone) on the investment of W dollars for L years at an annual interest rate r. It assumes that the amount W is paid off at the rate W/L per year continuously over the life of the equipment and is paid off completely at time L. This assumption may not be justified in practice, but alternatives to it would be equally suspect. In any case, Equation (47) is quite commonly used for costing purposes and is very easy to apply.

Again, it is desired to express I on a per-hour basis and relate it to specific units of equipment. Assume $r = .065$. No justification is given for this rate; it may, however, be regarded as the rate paid to bondholders by the firm. Then assuming L years

[26]This procedure for allocating depreciation costs to the linehaul process implicitly assumes that for depreciation purposes an hour of linehaul operations is the same as an hour of repair, assembly, terminal or other non-linehaul operations. Although this is surely not true in practice, no attempt is made in the text to refine the procedure of allocating depreciation expense to linehaul operations.

equipment life and using the locomotive purchase price equation, Equation(44), the per-hour interest cost allocable to a locomotive unit, I_L, is given by

$$I_L = \frac{(.5)\ (\$94HP_L - \$4313)\ (L)\ (r)}{8760L} = \$.000351HP_L - \$.016003\ . \qquad (48)$$

Similarly, using the purchase price in Table 6, the hourly interest cost of a standard box car, I_c, is given by

$$I_c = \frac{(.5)\ (\$13{,}083)\ (L)\ (r)}{8760L} = \$.048367\ . \qquad (49)$$

Thus the total interest cost of equipment to be allocated to a trip of T hours is

$$I_T = (I_L\mathcal{N}_L + I_c\mathcal{N}_c)\,T\ , \qquad (50)$$

where I_T = total interest cost of equipment for the trip
$\mathcal{N}_L$ = number of locomotive units
$\mathcal{N}_c$ = number of cars.

In summary, per-hour estimation equations for crew expenses, fuel expense, depreciation, and interest have been derived above. Therefore, to determine the cost of a particular train trip, one needs to know the trip time in hours as well as the hourly cost components. The product of trip time and hourly cost gives total trip cost. Note that trip cost is a function of (1) trip time, (2) number of cars, (3) number of locomotives, and (4) horsepower of each locomotive. See Equations (39, 43, 45, 46 and 50).

In addition, recall that it is assumed that the train operates at maximum speed, V_f, and encounters no delays en route. This implies that trip time, T, is equal to the minimum time in which the trip can be made, defined above as T_f. Trip time, T_f, may be determined from knowledge of V_f, given by the solution to Equation (33), and the trip distance D, in miles, as

$$T_f = D/V_f. \qquad (51)$$

Therefore, T_f is a function of (1) trip distance, (2) locomotive horsepower, (3) number of locomotives, (4) number of axles on each locomotive, (5) gradient of terrain, (6) degree of curvature, (7) number of cars, (8) number of axles on each car, (9) loading percentage of each car, and (10) load limit of each car. See Equation (37) above. Given these relationships, the cost behaviour of trains under a variety of circumstances may be investigated.

Cost Curves for Linehaul Process

Given the cost estimation equations of the preceding subsection and the train linehaul process function, it is possible now to investigate the cost behaviour of a train in the process of hauling freight between two points. Specifically, it is desired to develop short-run and long-run unit cost curves for the freight train. Unit cost will be measured in dollars per ton-mile and will be composed of all major linehaul expenses, including interest and depreciation on equipment (but no distribution of administrative expenses is included). The output measure is the ton-mile per hour. A "plant" consists of a locomotive of given horsepower; output varies in the short run with the number of cars comprising the train. In the long run the locomotive horsepower as well as the number of cars can vary. This is expressed by a family of short-

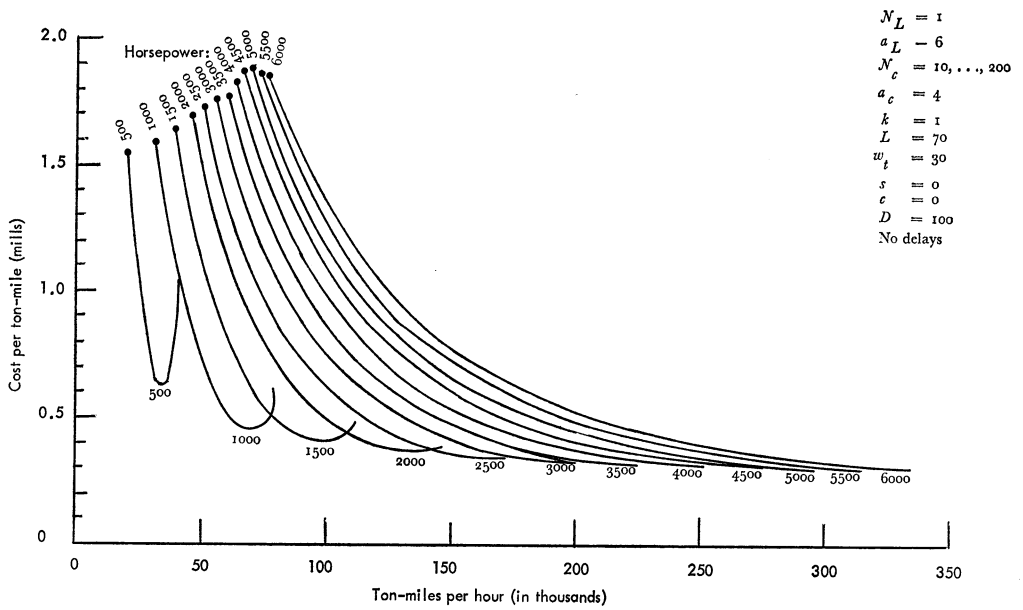

Figure 3. *Rail Linehaul Cost Curves*

run unit cost curves, each representing a given horsepower. The implied envelope curve to this family of short-run cost curves represents the long-run unit cost curve of the linehaul process.

Figure 3 displays a set of short-run average cost curves for the linehaul operation of a freight train. The assumptions under which these curves were obtained are: (1) trip distance of 100 miles; (2) zero gradient of terrain; (3) zero track curvature; (4) no delays en route; (5) one 6-axle engine unit; (6) identical, fully loaded 4-axle box cars; (7) weight-limited commodity; and (8) costs given by Equation (39) and Table 5, Equations (43, 45, 46, 48, and 49). The assumed physical characteristics of the trip are summarized in Figure 3.

Figure 3 presents the cost per ton mile of the assumed trip as horsepower and number of cars change. These were obtained by expressing trip cost on a per-hour basis and dividing the result by the appropriate ton-mile per hour figure. Each curve represents a given horsepower locomotive pulling between 10 and 200 cars. Each curve may be thought of as a "plant" unit cost curve, where horsepower represents the fixed input and number of cars represents the variable input. The unit cost curves for 500–3,000 horsepower display the U-shape usually assumed to characterize plant unit cost curves; but beyond 3,000 horsepower the curves are continually downward sloping. This implies that trains substantially longer than 200 cars would be required to make the plant unit cost curves rise. An implied envelope curve (not drawn), representing the long-run average cost of the train trip, clearly indicates economies of scale throughout the range of output from zero to 196,500 ton-miles per hour (a horsepower range up to 3,000). Economies of scale, in fact, seem to persist

throughout the range of output shown on the graph, although at a much diminished rate beyond an output of 196,500 ton-miles per hour.[27]

Curves such as those of Figure 3 can be derived for any proposed trip by specifying the necessary values of the appropriate variables. The results could be used in the selection of equipment for various train trips. In addition the linehaul costs generated via the process function could be used with other cost determination methods for the remaining rail processes to estimate the rail firm's costs of a train trip.

APPENDIX

Of the three possible sets of roots that may be obtained from cubic equations, the set resulting from the solution of Equation (33) will contain one real, positive root. That this is so may be seen from examination of the tractive-effort and drag-force functions. The tractive-effort function is a rectangular hyperbola; see Equation (20). The drag-force function, Equation (31), is quadratic in velocity: it reaches a minimum value at a negative velocity and proceeds into the first quadrant positively sloped.

This can be demonstrated as follows. Write the drag-force function in the general form of Equation (15):

$$F_d = A_0 + A_1 v + A_2 v^2,$$

where $A_0 = (1.3 + 20s + .8c)\ W + 29A > 0$

$A_1 = .03 W_L + .045 W_c > 0$

$A_2 = .29125 N_L + .04375 N_c > 0.$

Then $dF/dv = A_1 + 2A_2 v$. Setting this equal to zero and solving gives a first-order condition for an extremum as $v = -A_1/2A_2$, which is negative. Second-order condition is $d^2F/dv^2 = 2A_2$, which is positive, ensuring a minimum and ensuring that $F_d\ (v)$ is positively sloped to the right of its minimum. Graphically, these results may be shown as in Figure 4.

These characteristics of the two functions ensure an intersection in the first quadrant (therefore a real, positive root). The other two roots may be conjugate imaginary; real, negative, and unequal; or real, negative, and equal.[28]

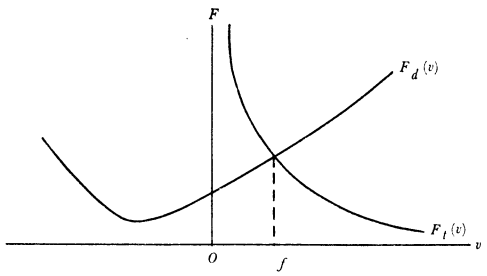

Figure 4. *Determination of Cruising Speed of Train*

[27]If delays en route are not dependent on variables entering the process function, delay time would merely result in a vertical shift in the cost curves. If, on the other hand, delays are a function of one or more variables affecting the linehaul process function, then not only the height but also the shapes of the cost curves might be affected. For example, if linehaul delays were positively related to train length the unit cost curves would all be at higher levels than shown in Figure 3, and would probably reach their minimum earlier and turn up more steeply.

[28]See Selby and Girling [18, p. 393].

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